

Cauchy's Arm Lemma on a Growing Sphere

Zachary Abel* David Charlton[†] Sébastien Collette[‡] Erik D. Demaine^{§¶}
Martin L. Demaine[§] Stefan Langerman^{||} Joseph O'Rourke^{**} Val Pinciu^{††}
Godfried Toussaint^{‡‡}

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Abstract

We propose a variant of Cauchy's Lemma, proving that when a convex chain on one sphere is redrawn (with the same lengths and angles) on a larger sphere, the distance between its endpoints increases. The main focus of this work is a comparison of three alternate proofs, to show the links between Toponogov's Comparison Theorem, Legendre's Theorem and Cauchy's Arm Lemma.

1 Introduction

A chain is composed of a sequence of points v_0, \dots, v_n , where each consecutive pair is connected by edges $e_i = v_{i-1}v_i$, and the angle at vertex v_i between e_i and e_{i+1} is θ_i . A chain is convex if the corresponding polygon (that is, if we add the edge v_0v_n to close the chain) is convex. These are well defined in the plane as well as on the sphere. In the case of the sphere, the edges are arcs of great circles. Cauchy's lemma can be expressed in the following way (although this was not its original statement):

Lemma 1 (Cauchy). *Given a convex chain C , if we increase the value of some nonempty subset of the angles θ_i , while keeping the length of the edges fixed and every $\theta_i \leq \pi$, then the distance between the endpoints v_0 and v_n strictly increases.*

*Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA. zabel@fas.harvard.edu

[†]Department of Computer Science, Boston University, 111 Cummington Street, Boston, MA 02215, USA. charlton@cs.bu.edu

[‡]Chargé de recherches du F.R.S.-FNRS, sebastien.collette@ulb.ac.be. Computer Science Department, Université Libre de Bruxelles, CP212, Bvd. du Triomphe, 1050 Brussels, Belgium

[§]MIT Computer Science and Artificial Intelligence Laboratory, 32 Vassar Street, Cambridge, MA 02139, USA, {edemaine, mdemaine}@mit.edu

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^{||}Chercheur qualifié du F.R.S.-FNRS, stefan.langerman@ulb.ac.be. Computer Science Department, Université Libre de Bruxelles, CP212, Bvd. du Triomphe, 1050 Brussels, Belgium

^{**}Department of Computer Science, Smith College, Northampton, MA 01063, USA, orourke@cs.smith.edu

^{††}Department of Mathematics, Southern Connecticut State University, USA, pinciu@scsu.ctstateu.edu

^{‡‡}School of Computer Science, McGill University, 3480 University St., Montreal, Canada, godfried@cs.mcgill.ca

Although the original proof was flawed, this lemma was later proved correctly in the plane, as well as when the chain is on the surface of a sphere [1, p. 228].

In this note, we consider the following question: what happens to the endpoints of a chain on a sphere if, instead of choosing some angles to increase, we change the *radius* of the sphere while preserving both the edge lengths and the angles?

Spherical geometry tells us that when convex polygons are drawn on the sphere, the sum of their angles is larger than in the convex polygons with same edge lengths in the plane. This increment is called the *spherical excess*. Its value decreases when the radius of the sphere grows, and also depends on the area of the polygon. Intuitively, as the sum of the angles decreases when the sphere gets larger, one might think that, given a convex chain C on the sphere S and a chain with same angles and edge lengths C' on a larger sphere S' , the endpoints of C' should be farther from each other than those of C .

However, this intuition alone does not suffice to reach this conclusion, because even though the sum of the angles is larger in C , it is not certain that every angle individually grows. This is what we show in the next sections, by analyzing the behavior of the angles of a triangle when drawn on different kinds of surfaces.

Most results in this note follow from a theorem in Riemannian geometry, Toponogov's Comparison Theorem [4], which we discuss in Section 4. Toponogov's version is strictly more general as it covers arbitrary manifolds with positive curvature, of which our "growing sphere" version is a special case.

However, we believe that the comparison of these approaches is of general interest because (1) the available proofs of Toponogov's Theorem rely on machinery in Riemannian geometry, whereas our proof is elementary; and (2) the statement of Cauchy's Arm Lemma in the alternate form provided here might be useful in further research directions.

2 Cauchy's Lemma from the Sphere to the Plane

In this section, we compare a chain drawn on a sphere to a similar chain drawn in the plane. Note that the plane is equivalent to a sphere of infinite radius, so this is clearly a subproblem of the original question. Our result is formally stated as follows:

Theorem 2. *Let C be a convex chain embedded on the sphere S , and C' be a convex chain with same angles and edge lengths in the plane. Then the distance between the endpoints of C' is greater than the distance between the endpoints of C .*

To derive this result, we first prove that given a triangle on the sphere S and a triangle with the same edge lengths in the plane, the latter has smaller angles. A more general proof of this will be provided in the next section. We believe, however, that this proof is interesting as it involves neither Legendre's Theorem nor Toponogov's Comparison Theorem.

Lemma 3. *Let $\triangle abc$ be a geodesic triangle on a sphere, with side lengths A , B , and C opposite vertices a , b , and c respectively. Let b' and c' be the midpoints of ab and ac respectively. Then the length $A' = |b'c'|$ of this "midchord" is strictly greater than $A/2$.*

Note that, in the plane, we would have $A' = A/2$.

Proof. Let the given triangle have angles α, β, γ at vertices a, b', c' respectively. Extend the geodesic segment $b'c'$ to b'' so that $|c'b''| = A'$, and connect b'' to c . See Figure 1. Then $\triangle cb''c'$ is congruent

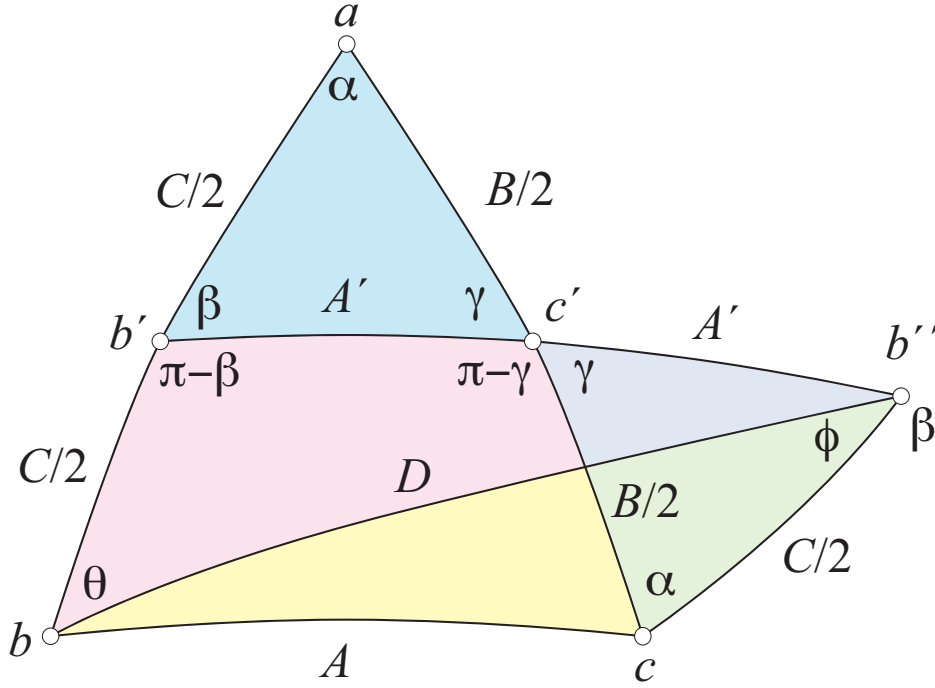


Figure 1: $\triangle cb''c'$ is congruent to $\triangle ab'c'$, and $\theta > \phi$.

to $\triangle ab'c'$, as it has the same angle γ at c' , included between the same side lengths A' and $B/2$.

Now draw the geodesic diagonal bb'' , and call the angles on its opposite sides θ and ϕ as illustrated. For the upper triangle $\triangle bb'b''$, we have $\theta + (\pi - \beta) + (\beta - \phi) > \pi$ (because the angles of a spherical triangle sum to strictly greater than π), i.e., $\theta > \phi$. Now notice that the two sides of the upper triangle determining θ have lengths D and $C/2$, and the two sides of the lower triangle determining ϕ have the same lengths D and $C/2$. Therefore, $\theta > \phi$ implies the same inequality in the lengths of the opposite sides: $2A' > A$. This establishes the claim of the lemma. \square

Lemma 4. *Let a planar triangle have side lengths A, B , and C . Each angle of the (unique) spherical triangle with the same side lengths is strictly larger than the corresponding planar angles.*

Proof. Let the planar triangle be $P = \triangle abc$ with angle α at a , and the spherical triangle be $S = \triangle a'b'c'$ with angle α' at a' . We prove that $\alpha' > \alpha$.

We draw midchords in both $P = P_0$ and $S = S_0$ as in Lemma 3, bisecting the sides of length B and C . Call these midchords p_1 and s_1 in P_0 and S_0 respectively, and call the triangles above these midchords P_1 and S_1 respectively. See Figure 2. We know that $|p_1| = A/2$ because P is planar, and that $|s_1| > A/2$ by Lemma 3. Repeat the construction on P_1 and S_1 , with midchords p_2 and s_2 . We have $|p_2| = |p_1|/2 = A/4$, and $|s_2| > |s_1|/2 > A/4$. Continuing in this manner, for any i , we obtain $|p_i| = A/2^i$ and $|s_i| > A/2^i$.

Let $B_i = B/2^i$ and $C_i = C/2^i$ be the side lengths of the triangles P_i and S_i after i iterations. Note that these side lengths are the same by construction for the planar and spherical triangles. Applying

the law of cosines to the planar triangle, we have

$$\alpha = \arccos \left[\frac{1}{2} \left(B_i/C_i + C_i/B_i - |p_i|^2/(B_i C_i) \right) \right]$$

Note that the length ratios in this expression are independent of i , as all the triangles P_i are similar to the starting triangle P . As $i \rightarrow \infty$, the spherical triangles S_i approach planarity in the limit, so we have

$$\alpha' = \lim_{i \rightarrow \infty} \arccos \left[\frac{1}{2} \left(B_i/C_i + C_i/B_i - |s_i|^2/(B_i C_i) \right) \right]$$

Because the ratios $B_i/C_i = B/C$ and $C_i/B_i = C/B$ are constant, the only difference between the α and α' expressions occurs in the $|p_i|$ and $|s_i|$ factors. But we know that $|s_i| > |p_i|$ for all i , and indeed $|s_i| - |p_i| > |s_1| - |p_1| > 0$, so

$$\lim_{i \rightarrow \infty} |s_i|^2/(B_i C_i)$$

is strictly greater than the constant $|p_1|^2/(B_i C_i) = A^2/(BC)$. Because the arccos function is monotonically decreasing, this implies that $\alpha' > \alpha$, as claimed.

Repeating the argument for each angle of the triangle establishes the lemma. \square

Thus, when a planar triangle is drawn on a sphere (with the same side lengths), all angles increase, and correspondingly, when a spherical triangle is drawn in the plane, all angles decrease.

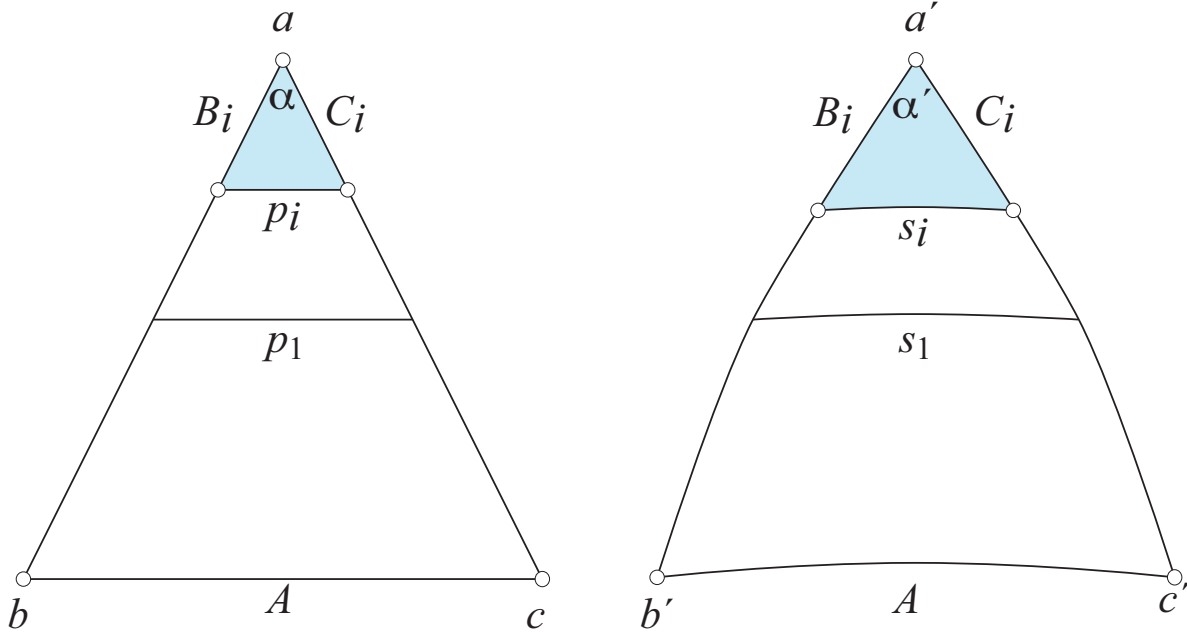


Figure 2: p_i is the i -th midchord in the planar triangle (left), and s_i in the spherical triangle (right). $|s_i| > |p_i|$ and $\alpha' > \alpha$.

Proof of Theorem 2. Starting with the convex chain C on the sphere, we triangulate it. We denote the set of triangles obtained by \mathcal{A} , which by the definition of a convex chain together form a convex polygon (see Figure 3). We look at \mathcal{A}' , the corresponding set of triangles in the plane, i.e, the set

of triangles with same edge lengths, and with adjacency preserved. This is always achievable, as the dual of a triangulation of a convex polygon is a tree.

We denote by C'' the chain in \mathcal{A}' corresponding to the chain C in \mathcal{A} . By Lemma 4, we know that all the angles of the triangles are smaller, and thus all the angles on the boundary of \mathcal{A}' are smaller. In particular C'' has same edge length as C , but every angle is strictly smaller; the endpoint of C and C'' are at the same distance from each other. Note that C'' is also a convex chain, because the convex angles of C only decreased, and therefore no reflex vertex can occur.

Recall that C' is a chain in the plane with same edge lengths and same angles as C . To transform C'' into C' , all the angles of C'' have to be increased. We apply Cauchy's Lemma in the plane, and deduce that the endpoints of C' are farther apart than those of C'' and thus than those of C . \square

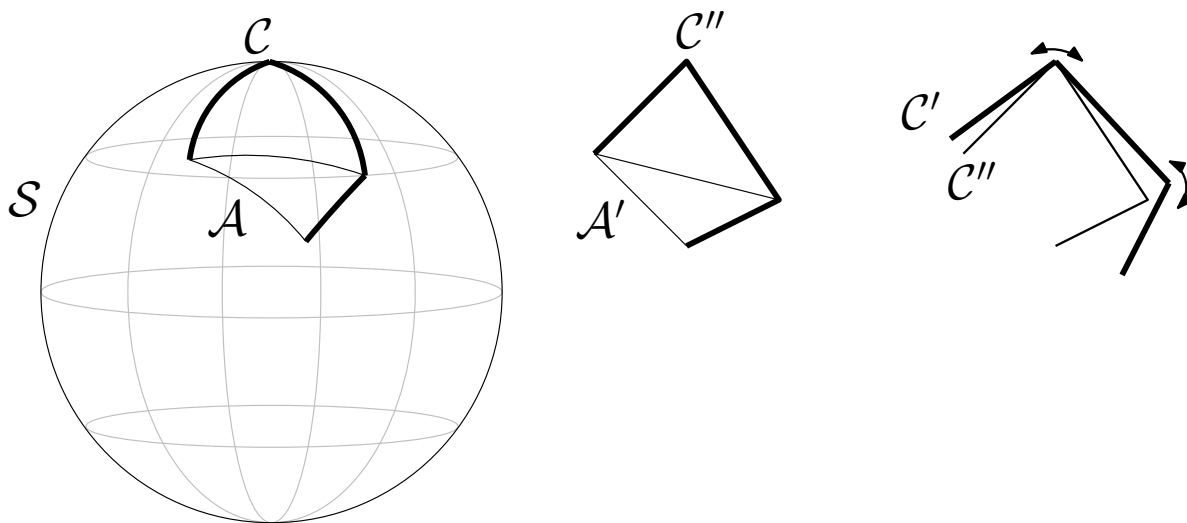


Figure 3: Illustration of the proof of Theorem 2.

3 Cauchy's Lemma on the Growing Sphere

In this section we generalize the result stated above to the case where we move from a sphere S to a larger sphere S' . To achieve this goal, we will make use of a theorem of Legendre dating back to 1798 [2]:

Theorem 5 (Legendre). *Given a triangle \mathcal{T} on a sphere S with angles $\theta_1, \theta_2, \theta_3$ and a triangle \mathcal{T}' in the plane with same edge lengths, and with corresponding angles $\theta'_1, \theta'_2, \theta'_3$, we have*

$$\forall i \in \{1, 2, 3\}, \theta'_i = \theta_i - \frac{\delta}{3} + (4)$$

where δ is the spherical excess of the triangle \mathcal{T} and (4) denotes a polynomial expression of the edge lengths of degree at least 4.

In other words, for sufficiently small triangles, that is, where the fourth order terms are negligible, the spherical excess is evenly split among the three angles. What the formula does not say,

however, is if, for all triangles on the sphere – even large ones – all three angles are *always* larger than their counterparts in the plane. A corollary of our results in the next section will show that this is actually the case.

Note that Legendre’s Theorem is also valid if we compare the same triangle drawn on two spheres \mathcal{S} and \mathcal{S}' of radius r and r' respectively, and with $r < r'$. In that case, its formulation is

$$\forall i \in \{1, 2, 3\}, \theta'_i = \theta_i - \frac{\delta - \delta'}{3} + (4)$$

where δ and δ' are the spherical excesses on both spheres. $(\delta - \delta')$ is always positive, as long as $r < r'$.

Here is our main Theorem:

Theorem 6. *Let C be a convex chain embedded on the sphere \mathcal{S} of radius r , and C' be a convex chain with same angles and edge lengths on the sphere \mathcal{S}' . Then the distance between the endpoints of C' is greater than the distance between the endpoints of C .*

To prove it, we proceed as follows, using two technical lemmas: first we show that given a thin triangle on a sphere \mathcal{S} , and a triangle with same edge lengths on a larger sphere \mathcal{S}' , the small angle is strictly smaller on \mathcal{S}' . Second, we generalize that to any triangle, showing that *every* angle is smaller on \mathcal{S}' . And finally, we combine these two results to prove our Theorem.

Lemma 7. *Let \mathcal{T} be a thin triangle embedded on the sphere \mathcal{S} of radius r , meaning that the apex angle $\theta_1 < \epsilon$ for some $\epsilon > 0$. Let \mathcal{T}' be a triangle with same edge lengths on the sphere \mathcal{S}' of radius $r' > r$. Then, $\theta_1 > \theta'_1$, where θ'_1 is the angle corresponding to θ_1 on the larger sphere.*

Proof. Given the radius r of the sphere \mathcal{S} , the parameter ϵ defines the largest possible length of the smallest edge in \mathcal{T} , i.e., the edge opposed to θ_1 . Let $\ell = f(\epsilon, r)$ be that maximal possible length. By adding Steiner vertices on the edges of \mathcal{T} , we can triangulate it such that every triangle has edges of lengths at most 2ℓ . We denote by \mathcal{A} the triangulation of \mathcal{T} .

Let \mathcal{A}' be the same set of triangles on \mathcal{S}' . As the spherical excess of every triangle on a sphere is positive, and because we can pick ϵ as small as needed – and thus have ℓ and the sides of our triangles as small as needed – the fourth order terms in Legendre’s Theorem are negligible. Thus, every angle of each triangle of \mathcal{A}' is strictly smaller than the corresponding one in \mathcal{A} . We also obtain that \mathcal{A}' is a convex region, just as in the proof of Theorem 2, because all the angles of the boundary of \mathcal{A} became smaller in \mathcal{A}' .

Now, consider the 2-chain \mathcal{L} composed of the two longer edges of \mathcal{T} . Let C be the chain on the convex hull of \mathcal{A} coinciding with \mathcal{L} , i.e., C is the subdivision of \mathcal{L} in a chain where all edges have length less than 2ℓ . Let C' be the corresponding chain on the sphere \mathcal{S}' , defined by the convex hull of the set of triangles \mathcal{A}' .

Starting with C' , we increase all the angles to obtain a 2-chain \mathcal{L}' , with same edge lengths and angles as in \mathcal{L} . Recall that Cauchy’s Lemma is valid on a sphere, so we know that the endpoints of \mathcal{L}' are farther apart than in C' . In other words, if we want to close the 2-chain \mathcal{L}' to obtain the triangle \mathcal{T}' with same edge length as in \mathcal{T} , we need to shrink the small angle. Thus, we conclude that $\theta_1 > \theta'_1$, which completes the proof. \square

Lemma 8. *Let \mathcal{T} be a triangle embedded on the sphere \mathcal{S} of radius r . Let \mathcal{T}' be a triangle with same edge lengths on the sphere \mathcal{S}' of radius $r' > r$. Then, $\theta_i > \theta'_i$ for $i \in \{1, 2, 3\}$, where θ'_i is the angle corresponding to θ_i on the larger sphere.*

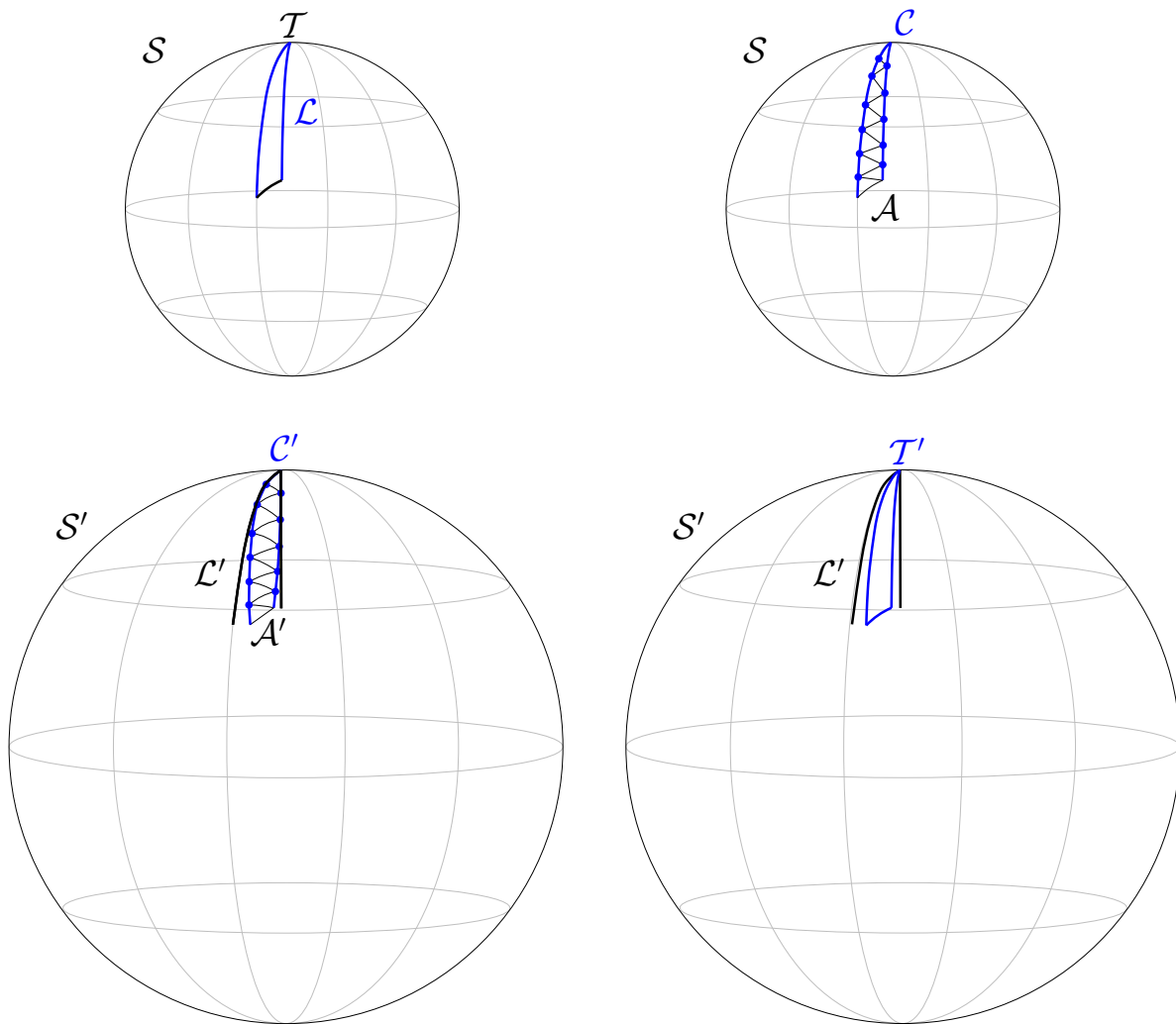


Figure 4: Illustration of the proof of Lemma 7.

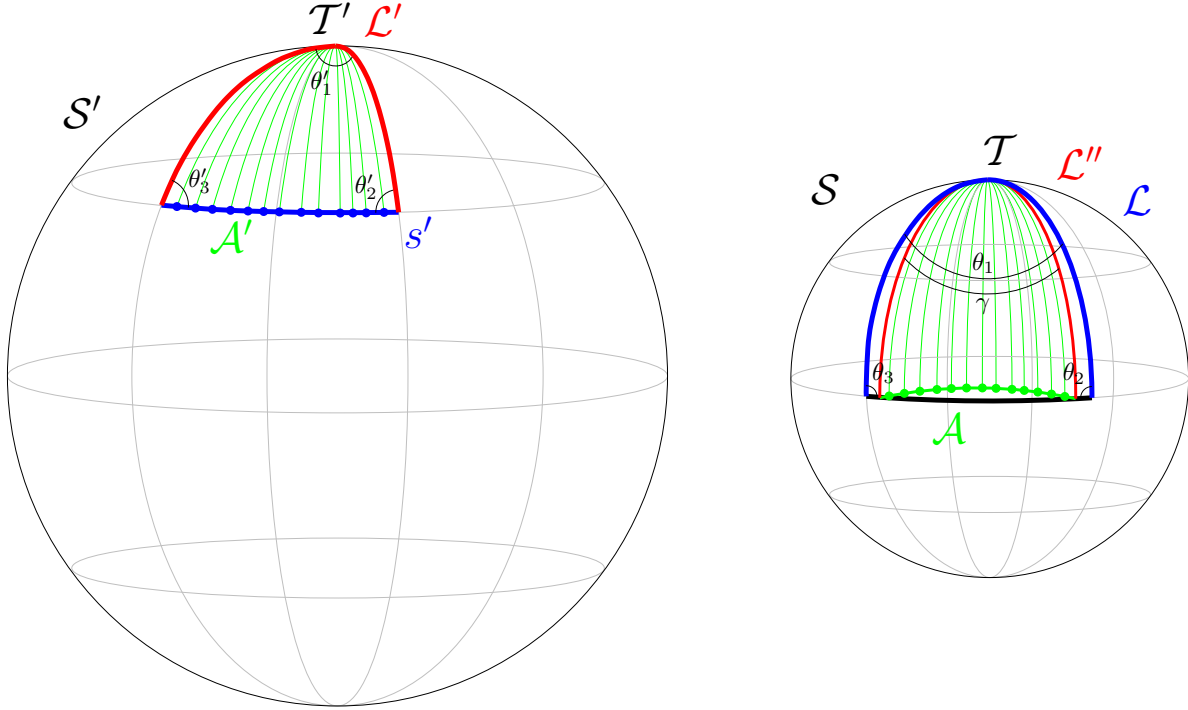


Figure 5: Illustration of the proof of Lemma 8.

Proof. Let s' be the side of \mathcal{T}' opposed to θ'_1 , and \mathcal{L}' denote the 2-chain composed by the two other sides. Decompose s' into as many subsegments as needed to make sure that each has length less than ℓ , which is a value depending on r and r'^1 . We denote by \mathcal{A}' the fan-triangulation based on these subsegments.

Let \mathcal{A} be the corresponding set of triangles on \mathcal{S} , with edge lengths preserved. Let \mathcal{L}'' be the 2-chain corresponding to \mathcal{L}' in \mathcal{A} , with angle γ . Note that s , the chain corresponding to s' on \mathcal{S} , is not straight, because by Lemma 7, we know that the small angle of every triangle is smaller on \mathcal{S}' than on \mathcal{S} . Also, we know that γ is strictly larger than θ'_1 , for the same reasons.

Because the sum of the length of the edges of s is exactly equal to the length of s' , and because every angle in the chain is strictly less than π , the distance between the two endpoints of \mathcal{L}'' is strictly smaller than s . This means that the angle θ_1 is strictly larger than γ . We conclude that $\theta'_1 > \theta_1$.

It only remains to note that we could have picked any of the three angles of \mathcal{T}' , and thus every angle is strictly larger in \mathcal{T} than in \mathcal{T}' . \square

We now have all the tools required to prove Cauchy's Lemma on the growing sphere.

Proof of Theorem 6. The proof is similar to that of Theorem 2. Starting with the chain C , we triangulate it, obtaining \mathcal{A} . We look at \mathcal{A}' the corresponding set of triangles on \mathcal{S}' . By Lemma 8, we know that all the angles of the triangles are smaller, and thus all the angles on the boundary of \mathcal{A}' are smaller. To make the boundary of \mathcal{A}' and the chain C' coincide, we have thus to increase all an-

¹We do not need to know ℓ precisely, it should just be small enough so that we can apply Lemma 7 later on.

gles. Using Cauchy's Lemma on the sphere (which applies by the same reasoning as in Theorem 2), we deduce that the endpoints of C' are farther apart than those of C . \square

In summary, when a convex chain on one sphere is redrawn (with the same lengths and angles) on a larger sphere, the distance between its endpoints increases.

4 Generalization to Complete Riemannian Manifolds of Positive Sectional Curvature

In this section we present a more general version of Lemma 8, and prove it by a simple application of Toponogov's Comparison Theorem [4]. Here is the part of the theorem relevant to our application [3]:

Theorem 9 (Toponogov). *Let M be a complete Riemannian manifold² with sectional curvature³ $K \geq \kappa$. Given points p_0, p_1, q in M satisfying $p_0 \neq q, p_1 \neq q$, a non constant geodesic⁴ c from p_0 to p_1 and minimal geodesics c_i , from p_i to q , $i = 0, 1$, all parameterized by arc length. Suppose the triangle inequality $|c| \leq |c_1| + |c_2|$ is satisfied and $|c| \leq \frac{\pi}{\sqrt{\kappa}}$ in the case $\kappa \geq 0$. $\alpha_i \in [0, \pi]$ denote the angles at p_i , $\alpha_0 = \angle(\dot{c}_0(0), \dot{c}(0))$, $\alpha_1 = \angle(\dot{c}_1(0), -\dot{c}(|c|))$. Then there exists a corresponding comparison triangle $\tilde{p}_0, \tilde{p}_1, \tilde{q}$ in the model space M_κ^2 with corresponding geodesics $\tilde{c}_0, \tilde{c}_1, \tilde{c}$ which are all minimal of length $|\tilde{c}_i| = |c_i|, |\tilde{c}| = |c|$ and the corresponding angles $\tilde{\alpha}_i$ satisfy $\tilde{\alpha}_i \leq \alpha_i$.*

The interesting part for the application we have in mind is to notice that the two spheres we consider in the previous sections are actually model spaces M_κ^2 , with κ depending on the radius of the sphere. So, we obtain directly a generalized version of the lemma used in previous sections:

Corollary 10. *Let \mathcal{T} be a triangle embedded on a complete Riemannian manifold M_κ with sectional curvature κ and where the triangle inequality holds. Let \mathcal{T}' be a triangle with same edge lengths on a model space $M_{\kappa'}^2$, with sectional curvature $\kappa' < \kappa$, also satisfying the triangle inequality. Then, $\theta_i > \theta'_i$ for $i \in \{1, 2, 3\}$, where θ'_i is the angle corresponding to θ_i on \mathcal{R}' .*

We can use this to get an alternate proof of Theorem 6, with the same arguments, using the model spaces M_κ^2 and $M_{\kappa'}^2$ which are two spheres of radius $r = \frac{1}{\sqrt{\kappa}}$ and $r' = \frac{1}{\sqrt{\kappa'}}$.

Open Problem 1. *What is the class of surfaces on which the original version of Cauchy's Arm Lemma (Lemma 1) is true?*

The answer to this question would provide the class of surfaces on which we could generalize Cauchy's Lemma, i.e., redrawing the same chain (angles and lengths preserved) on two different surfaces and deducing that the endpoints move farther apart. We can reuse the same reasoning as long as (the original version of) Cauchy's Arm Lemma and Toponogov's Theorem apply to the surfaces we consider.

²A Riemannian manifold is a differentiable manifold whose tangent spaces support a smoothly varying inner product, which permits angles and lengths to be defined. It is *complete* if Cauchy sequences have limits.

³The *sectional curvature* measures the deviation of geodesics, and is the natural generalization of Gaussian curvature for surfaces. A sphere has constant positive sectional curvature.

⁴A *non constant geodesic* is any geodesic of positive length.

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