Edge-Unfolding Orthogonal Polyhedra is Strongly NP-Complete

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Abstract

We prove that it is strongly NP-complete to decide whether a given orthogonal polyhedron has a (nonoverlapping) edge unfolding. The result holds even when the polyhedron is topologically convex, i.e., is homeomorphic to a sphere, has faces that are homeomorphic to disks, and where every two faces share at most one edge.

1 Introduction

An *edge unfolding* of a polyhedron consists of cutting the surface along a subset of its edges in such a way that the surface can be unfolded into one planar piece without overlap.[1] Edge unfoldings have a long history, dating back to Albrecht Dürer in 1525; see [3]. The most famous open question is whether every convex polyhedron has an edge unfolding, but nonconvex polyhedra are even more interesting for practical manufacturing applications. The theoretical study of such unfoldings began at CCCG 1998 [2] and CCCG 1999 [1]. Biedl et al. [2] found some orthogonal polyhedra with no edge unfoldings, but the examples had faces with holes or two faces that shared two edges. Bern et al. [1] found a triangulated polyhedron with no edge unfolding that is homeomorphic to a sphere, implying that the polyhedron is *topologically convex*—has the graph (1-skeleton) of a convex polyhedron. In the journal version of their CCCG 1999 paper [1], they asked for the computational complexity of deciding whether a given triangulated polyhedron has an edge unfolding.

In this paper, we settle the computational complexity of the closely related problem of deciding whether a topologically convex orthogonal polyhedron has an edge unfolding. Specifically, we prove this *Orthogonal Edge Unfolding* problem is strongly NP-complete.

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[1] We allow boundary edges to touch in unfoldings, requiring only that the interior of the cut surface does not overlap itself.

2 Unique Coordinate Square Packing

The Square Packing problem asks, given n squares $s_1, \ldots, s_n$ of side-lengths $a_1, \ldots, a_n$ and a target distance d, whether there is some (non-overlapping) orthogonal packing of the squares s_i into a square of side-length d. This is known to be strongly NP-complete [4]. We first show that we may impose a few simplifying assumptions on the packings produced by this problem:

Definition 1 (Unique Coordinate Square Packing) *An instance of the Unique Coordinate Square Packing (UCSP) promise problem has the form $(d, (a_1, \ldots, a_n))$, where all values are positive integers and $a_i \le d - 2$ for each $1 \le i \le n$. In a YES instance, there exists an orthogonal packing of n squares $s_1, \ldots, s_n$ of side-lengths $a_1, \ldots, a_n$ into the square $D = [0, d] \times [0, d] \subset \mathbb{R}^2$ satisfying the following additional properties:*

- *all vertices of all squares in the packing have integer coordinates,*
- *no two vertices of two different squares have the same x- or y-coordinate, and*
- *no square in the packing touches the boundary of D.*

In a NO instance, there does not exist any orthogonal packing of the s_i into D.

Theorem 2 *The Unique Coordinate Square Packing problem is strongly NP-hard.*

The simple but technical proof is omitted from this extended abstract.

3 Overview

This section provides an overview of the detailed constructions to follow.

We first consider the problem of unfolding orthogonal polyhedra *with boundary* in Section 4, proving hardness by reduction from a UCSP instance $(d, (a_1, \ldots, a_n))$. We construct a polyhedron B with boundary (Figure 3) involving n squares b_i with side-lengths a_i (call these "blocks") surrounded by filler material. The polyhedron is designed to force the blocks to unfold inside a "cage" of shape $d \times d$, such that an unfolding exists if and only if there exists a square packing. (In the construction below, the blocks and cage are scaled up by a large factor q.) As the unfolding must remain connected, we use thin "wires" made from the filler material to "wind" around the blocks and connect them to the boundary

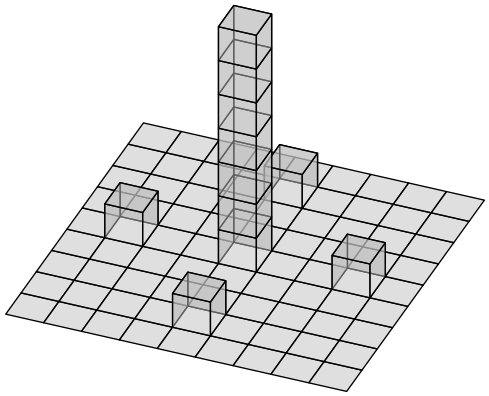


Figure 1: A depiction of an "atom," the polyhedral surface with 9×9 square boundary that enables universal wire unfolding as in Theorem 3. An atom is composed of 125 unit-square faces.

of the cage. The un-needed filler material winds itself out of the way. The univesally windable wires are described and proved in Section 4.1, and the details of the unfolding are presented in Sections 4.2, 4.3, and 4.4.

In Section 5 we reduce to orthogonal polyhedra without boundary by extending B into a polyhedron C with the property that C has an unfolding if and only if B does. This is accomplished with two U-shaped polygons (Figure 6) that must be separated from B to avoid overlap, which forces the extra material of C not to interfere with the unfolding of B.

4 Polyhedron With Boundary

In this section we show that the edge unfolding problem for an orthogonal polyhedron *with boundary* is NP-hard.

4.1 Atoms and Universal Wire Unfolding

As described in the overview, we require "winding wires" that can unfold into an arbitrarily chosen orthogonal path. We construct those here.

Define an ***atom*** as the polyhedral surface with boundary in Figure 1, whose boundary is a 9×9 square. The width of an atom is called the ***atomic width***, $w_A = 9$. Atoms are named thus since they are the basic "winding wire" unit, and also due to their tiny size relative to many constructions to follow.

For a finite or infinite grid G of $u \times u$ squares in the xy-plane and integers i, j, write $G[i,j]_u$ for the $(i,j)^{\text{th}}$ cell in G, i.e., the $u \times u$ cell positioned at (ui, uj). Similarly, if e is a directed line segment of length ℓ and u evenly divides ℓ, express e as the union of ℓ/u directed segments of length u and let $e[i]_u$ be the i^{th} such segment.

Define a ***wire W of length k in G*** as a simple path of connected squares in G: specifically, a collection of distinct squares $c_i = c_i(W)$ $(0 \le i \le k-1)$ in G and distinct, oriented edges $e_i = e_i(W)$ $(0 \le i \le k)$ such that e_i is the common edge of cells c_{i-1} and c_i for each $1 \le i \le k-1$, edge e_0 (the ***starting edge***) is an edge of the ***starting cell*** c_0, and e_k (the ***ending edge***) is an edge of the ***ending cell*** c_{k-1}. Edge e_i is oriented to trace the boundary of c_{i-1} clockwise, or equivalently, to trace the boundary of c_i counterclockwise. (Use the former condition for e_k and the latter for e_0.) It is convenient to discuss the ***medial path*** of a wire that connects the centers of $e_0, c_0, e_1, \ldots, c_{k-1}, e_k$ sequentially. The wire ***turns right, straight, or left*** at square c_i if the medial path turns right, straight, or left there.

If W is a wire of $w_A \times w_A$ squares in the x, y-plane, we can form the associated ***wire of atoms*** $A(W)$, a polyhedral surface with boundary, by replacing each square c_i with an atom a_i pointing in the positive z-direction such that atoms a_{i-1} and a_i are connected along the edge corresponding to e_i. Each unit-length edge $e_i[s]_1$ (for $0 \le s \le w_A - 1 = 8$) corresponds to an edge of one or two unit-square faces on $A(W)$.

Define a ***flatom***[2] as a $w_F \times w_F$ square where $w_F = 27$ is the ***flatomic width***. We will now show that wires of atoms can be universally unfolded in the following sense: roughly, any wire of k atoms can be unfolded inside any desired wire of k flatoms, while ensuring that the middle of each atom edge unfolds to the center of the corresponding flatom edge (or one unit away from center).

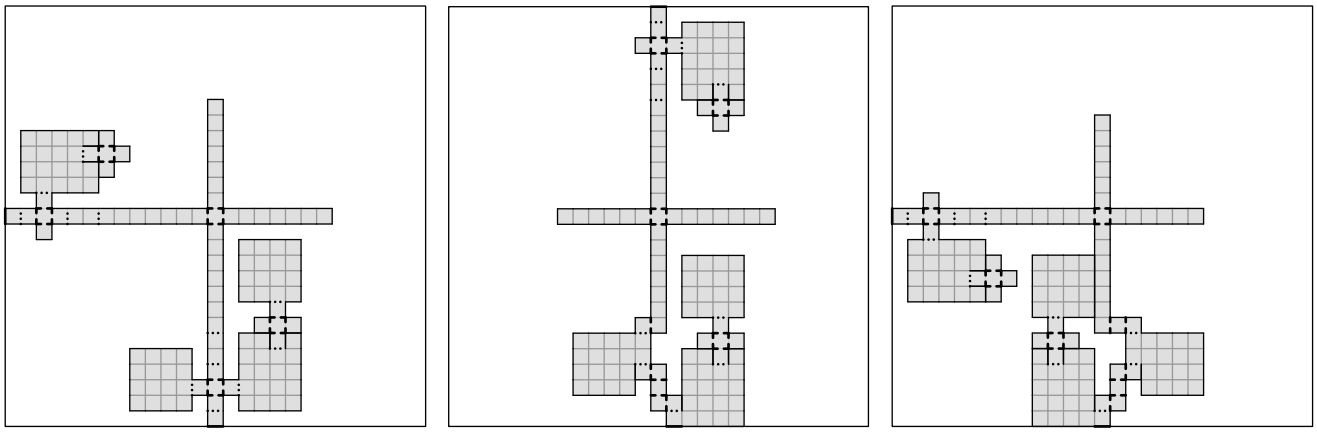
Theorem 3 *Let W and W' be any two wires of length k with side-lengths w_A and w_F respectively. Then for each $t \in \{12, 13, 14\}$ there is an edge unfolding of the wire of atoms $A(W)$ that lies inside W' such that $e_0(W)[4]_1$ (the middle unit edge of $e_0(W)$) unfolds to $e_0(W')[t]_1$ (i.e., the middle edge of $e_0(W')$ or one unit away) and $e_k(W)[4]_1$ unfolds to $e_k(W')[u]_1$ for some $u \in \{12, 13, 14\}$. Furthermore, this unfolding can be accomplished so that t and u have the same (resp., different) parity when W and W' together have an even (resp., odd) total number of left and right turns.*

Proof. By induction on k, it suffices to prove only the case $k = 1$. There are thus 27 cases: W turns right, straight or left; W' turns right, straight, or left; and $e_0(W)[4]_1$ unfolds to $e_1(W')[12]_1$, $e_1(W')[13]_1$, or $e_1(W')[14]_1$. We label these unfoldings of an atom by a quadruple $[X, Y, t, u]$, where $X, Y \in \{\text{L}, \text{S}, \text{R}\}$[3] indicate the directions of the turns of wires W and W' respectively, and t and u are as above. We must show that each of the 27 tuples (X, Y, t) appears in some unfolding $[X, Y, t, u]$ with the required parity constraints on t and u. Up to mirror-reflection and direction reversal, only ten unfoldings are required[4]. Three of these are

[2] short for "flat atom"

[3] These are abbreviations for Left, Straight, and Right turns.

[4] For example, these ten suffice: $[\text{L}, \text{L}, 13, 13]$, $[\text{L}, \text{L}, 14, 12]$,



(a) [L, L, 13, 13]. (b) [L, S, 14, 13]. (c) [R, L, 13, 13].

Figure 2: Three of the ten required unfoldings of an atom inside a flatom. Each unfolding is labeled with $[X, Y, t, u]$ as described in the proof of Theorem 3. Solid black lines indicate cuts, dotted lines are valley folds, and dashed lines are mountain folds. Gray lines are uncreased edges.

illustrated in Figure 2, with the remaining seven to be included in the full version. □

4.2 The Construction

Here we specify the polyhedron with boundary used in the reduction. The remainder of Section 4 is devoted to proving its correctness.

It will be useful to package atoms into a ***molecule***: a 2×2 grid of atoms whose boundary is a $w_M \times w_M$ square, where $w_M = 2w_A = 18$. Much of the reduction below uses a molecule as a basic unit of construction.

Begin with a Unique Coordinate Square Packing instance $(d, (a_1, \ldots, a_n))$. Define $q = 2^5 \cdot 3^4 \cdot nd$ (a large scale factor), and let $q_M = q/w_M$ be the number of molecules that fit across a distance q. Also set $t = (n+1+a_1+\cdots+a_n)q_M$, and $p = 500(4dq_Mt+t^2)+3w_A$; these choices will be explained shortly. Define the polyhedron with boundary $B(d, (a_1, \ldots, a_n))$ as the surface shown in Figure 3, to be described in more detail presently. The diagram is oriented so that the positive x and y directions are right and up respectively, and z is out of the page.

The face F_{floor} in Figure 3a is a single polygon formed by creating a $w_F \times p$ hole, H_{drain}, and a $dq \times dq$ hole, H_{cage}, in a large square of size $\ell = p + dq + w_A$. The two faces F^1_{pipe} and F^2_{pipe}, of widths $2w_A$ and w_A respectively, exactly fill H_{drain}. Five (not flat!) polyhedral surfaces $T_{\text{bottom}}, T_{\text{left}}, T_{\text{mid}}, T_{\text{right}}, T_{\text{top}}$, shown in detail in Figure 3b, form the sides of the ***tower***, T, which connects along the boundary of H_{cage}. The polyhedral surface T_{bottom}, whose boundary is a $dq \times tw_M$ rectangle, is a $dq_M \times t$ grid of molecules facing away from the tower except for the n square faces $b_1, \ldots, b_n$—called ***bricks***—of side-lengths $qa_1, \ldots, qa_n$, where brick b_i is positioned at $(q, (i + a_1 + \cdots + a_{i-1})q)$ relative to the bottom-left corner of T_{bottom}. (For T_{bottom}, "right" and "up" refer to the positive x and z directions, respectively, as in Figure 3b.) The parameter t was chosen so that these bricks exactly fit with q separation from each other and from the bottom and top edges. Recall that $a_i \le d-2$ for each i, so there is at least q separation between each brick and the right edge of T_{bottom}. The other four sides of T, which have dimensions $dq \times tw_M$ or $dq \times dq$, are completely tiled with outward-facing molecules.

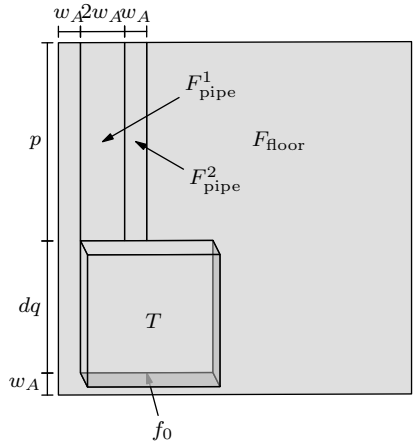
A single molecule has surface area 500, so the total surface area of T is strictly less than what the surface area would be if each brick were also tiled with molecules, namely $500(4dq_Mt+t^2) < p-3w_A$. Furthermore, the height of a molecule (out of the plane of its boundary) is 7, so the projection of T onto the plane containing F_{floor} extends beyond H_{cage} by only seven units. In particular, this projection lies strictly in the interior of the bounding box of F_{floor}, and is at least $p - 7 > w_A$ units away from the top edge of F_{floor}.

We will show in the next two subsections that $B(d, (a_1, \ldots, a_n))$ has an edge unfolding if and only if $(d, (a_1, \ldots, a_n))$ is a YES instance of UCSP. One direction is straightforward:

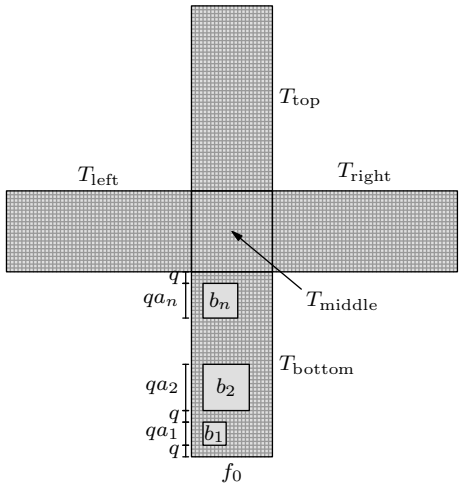
Lemma 4 *If $(d, (a_1, \ldots, a_n))$ is a UCSP instance and $B(d, (a_1, \ldots, a_n))$ has an edge unfolding, then $(d, (a_1, \ldots, a_n))$ is a YES instance.*

Proof. Fix some unfolding of $B = B(d, (a_1, \ldots, a_n))$. Let F^{1*}_{pipe} be the bottom height-1 subrectangle of F^1_{pipe}, and similarly for F^{2*}_{pipe}, and consider the polyhedral surface B^* obtained by replacing F^1_{pipe} and F^2_{pipe} with F^{1*}_{pipe} and F^{2*}_{pipe}. The unfolding of B induces an unfolding of

[L, S, 12, 13], [L, S, 14, 13], [S, L, 14, 13], [S, L, 12, 13], [S, S, 13, 13], [S, S, 12, 12], [R, L, 13, 13], [R, L, 14, 12].



(a) The global structure of surface B. Faces F_{floor}, F^1_{pipe}, and F^2_{pipe} are each a single polygon, but tower T is mostly covered with molecules as detailed in part (b). For ease of viewing, the image here is not drawn to scale: the width of the two pipes is significantly smaller than the width of the tower T, for example.



(b) A detail of the tower, T. Each surface of T is entirely tiled with molecules except for T_{bottom}, which has bricks $b_1, \ldots, b_n$—each a single square face—arranged as shown.

Figure 3: Detailed depiction of polyhedral surface B.

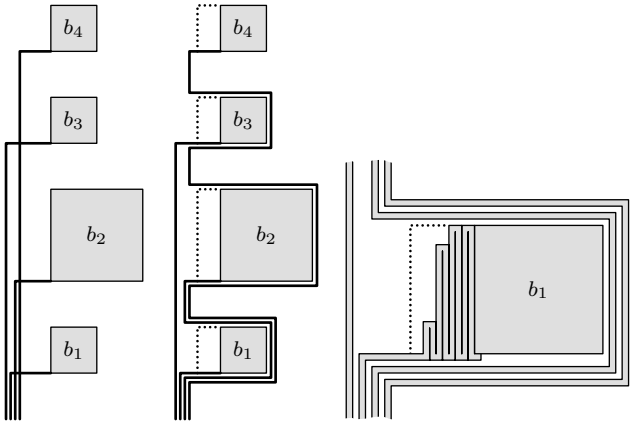
B^*. In this unfolding of B^*, all of $T \cup \{F^{1*}_{\text{pipe}}, F^{2*}_{\text{pipe}}\}$ unfolds into $H_{\text{cage}} \cup H_{\text{drain}}$: indeed, p was chosen to ensure that the surface area of $T \cup \{F^{1*}_{\text{pipe}}, F^{2*}_{\text{pipe}}\}$ is strictly less than p, so there is not enough material to reach the top of H_{drain}. It follows that each brick b_i unfolds into $H_{\text{cage}} \cup H_{\text{drain}}$, and since H_{drain} is too narrow for the bricks, each b_i unfolds into H_{cage}. So there exists a packing of the b_i (with side-lengths $a_i \cdot q$) into H_{cage} (with side-length $d \cdot q$), which proves the Lemma. □

4.3 Wiring the Tower

Think of T_{bottom} as a grid of molecules, with origin $(0, 0)$ at its lower left corner. In this section we demonstrate how to connect each brick to the bottom-left corner of the tower by a chain of molecules. For convenience, we ensure all such chains have the same length, L.

Brick b_i is positioned at $(q, y_i q)$ where $y_i = i + a_1 + \cdots + a_{i-1}$. Let f_i $(1 \le i \le n)$ be the lower edge of b_i, oriented left-to-right, and let f_0 be the lower edge of T_{bottom}, also oriented left-to-right. For $1 \le i \le n$ define $u_i = T_{\text{bottom}}[6i, 0]_{w_M}$; these are lined along the left of f_0 in T_{bottom}, spaced 6 molecules apart. Also let $v_i = T_{\text{bottom}}[q, y_i - 1]_{w_M}$ be the molecule just under the lower left corner of b_i in T_{bottom}.

Lemma 5 *For any permutation σ of $\{1, \ldots, n\}$, there exist n non-overlapping wires $W_1, \ldots, W_n$ of molecules in T_{bottom} such that each wire W_i has length exactly $L = 4ndq_M$ and connects $c_0(W_i) = u_i$ to $c_L(W_i) = v_{\sigma i}$, with starting and ending edges along f_0 and $f_{\sigma(i)}$ respectively. Furthermore, no wire touches the two leftmost columns of molecules on T_{bottom}, and finally, the complement of the bricks b_i and wires W_i in T_{bottom} forms a single edge-connected polyomino of molecules.*



(a) In the first step, overlapping wires are drawn from u_i to $v_{\sigma(i)}$ with just one right turn.

(b) Wires are modified with detours around bricks to avoid intersections.

(c) Finally, each wire W_i is modified with zig-zags in the empty $a_{\sigma(i)}q_M/2 \times q_M/2$ grid next to brick $b_{\sigma(i)}$ in order to bring its length up to exactly $L = 4ndq_M$.

Figure 4: The three steps in the construction of molecule wires W_i of Lemma 5. The figures correspond to $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 2$, and $\sigma(4) = 4$.

Proof. Provisionally define each W_i as the wire that goes straight up from u_i and turns right to $v_{\sigma(i)}$, as in Figure 4a. As defined, these wires may intersect: the horizontal segment of W_i hits the vertical segment of W_j when $i < j$ but $\sigma(i) > \sigma(j)$. To fix these, for each i, take all wires W_j that hit the horizontal part of W_i and insert a detour around brick b_i as illustrated in Figure 4b, keeping a 1-molecule gap between two detouring wires, and between these wires and b_i. Because $q_M > 4n$, there is ample room for the detours. Before the detours, each wire had length less than

$t+q_M = (n+2+a_1+\cdots+a_n)\,q_M \le 2ndq_M$, and each of fewer than n detours adds at most $2dq_M$ molecules, so the total length of each W_i is less than $4ndq_M$. Furthermore, by the parity of the positions of u_i and $v_{\sigma(i)}$, W_i has even length.

We now bring the length of each W_i up to exactly $L = 4ndq_M$. The $a_i q_M \times \frac{q_M}{2}$ grid of molecules to the left of b_i is empty, its bottom edge is adjacent to wire W_i, and the top and left edges are not adjacent to any wires. This grid has even width $q_M/2$, and by zig-zagging up and down in this region as shown in Figure 4c, we can add any even number of molecules up to $a_i q_M^2/2 > L$. This indeed allows each wire to reach its destination with total length exactly L. Finally, the left two columns of molecules were not touched by the wires, and the 1-molecule gaps inserted above ensure that the complement of the bricks and wires remains connected. □

Now think of T (partially unfolded as in Figure 3b) as a grid of atoms, not molecules. Edges $f_0, f_1, \ldots, f_n$ are as defined above, and let g be the bottom edge of $F^1_{\text{pipe}} \cup F^2_{\text{pipe}}$ of length $3w_A$, oriented left to right.

Lemma 6 *It is possible to write T as an interior-disjoint union of the following pieces:*

- *bricks $b_1, \ldots, b_n$,*
- *wires $X_1, \ldots, X_n$ of atoms where each X_i has length exactly $4L$ and connects the bottom-right corner of molecule u_i (with starting edge along f_0) to the top-right corner of molecule $v_{\sigma(i)}$ (with ending edge along $f_{\sigma(i)}$), and*
- *a wire X_0 connecting $(T_{bottom}[1,0]_{w_A}, f_0[1]_{w_A})$ to edge $g[1]_{w_A}$ along with its adjacent atom on T_{top}.*

Proof. Let wires of molecules W_i be as in Lemma 5. Wire X_i is obtained from W_i by starting at the bottom-right atom in molecule u_i and ensuring that $c_{4k}(X_i), \ldots, c_{4k+3}(X_i)$ are the four atoms in molecule $c_k(W_i)$ for each $0 \le k \le L-1$. This can be done uniquely, and by parity, this wire X_i will terminate at the top-right atom of molecule $v_{\sigma(i)}$.

It remains to construct X_0. Let W_0 be the wire of molecules in T that starts at $T_{\text{bottom}}[0,0]_{w_M}$ and traces the left edge of T_{bottom}, the bottom, left, and top edges of T_{left}, and the left edge of T_{top} up to its top-left corner. Let X_0' be the length-four wire of atoms that traces $c_0(W_0)$ as in Figure 5, and define X_0'' as the wire of atoms that follows the rest of W_0 as in the Figure. Let G be the region of T outside of the bricks $b_1, \ldots, b_n$ and wires $X_1, \ldots, X_n, X_0', X_0''$; by Lemma 5, G forms a connected polyomino of molecules. Lemma 5 guarantees that molecules $G[1,0]_{w_M}$ and $G[1,1]_{w_M}$ are in G, so pick any spanning tree S of the molecules in G in which these two molecules are connected. The desired wire of atoms X_0 is obtained by traversing X_0', walking all the way around S to the starting edge of X_0'', and then following X_0''. □

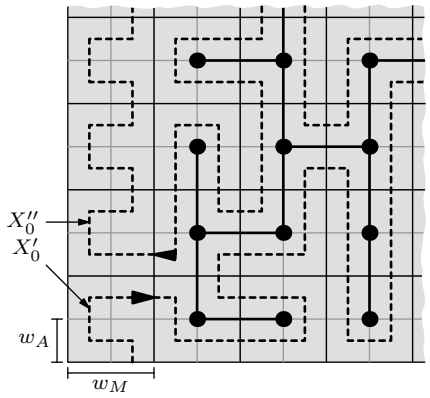


Figure 5: A closeup of the bottom-left corner of T_{bottom} illustrating how to write G as a wire of atoms as in the proof of Lemma 6. This wire, X_0, is formed by traversing the four atoms of wire X_0', then walking around the spanning tree S, and finally following atom-wire X_0''.

4.4 Unfolding Surface B

We are now able to prove the converse of Lemma 5:

Lemma 7 *If $(d, (a_1, \ldots, a_n))$ is a YES instance of UCSP, then $B(d, (a_1, \ldots, a_n))$ has an edge unfolding.*

Proof. Think of H_{cage} as a $dq_F \times dq_F$ grid of flatoms, where $q_F = q/w_F$, with origin in the lower-left corner. Let f_0 and f_i $(1 \le i \le n)$ be the bottom edges of H_{cage} and brick b_i respectively, as above. Pick a packing of squares with side-lengths $a_1, \ldots, a_n$ into $[0,d]^2$ with all the guarantees of the YES-promise of UCSP, and say the i^{th} square is positioned at (x_i, y_i). Scale this up to a packing of bricks b_i into H_{cage}, with b_i positioned at $(x_i q_F w_F, y_i q_F w_F)$. Since the bricks b_i do not meet each other or the edges of H_{cage}, there is at least a q_F-flatom separation between them. For $1 \le i \le n$, define the flatoms $h_i = H_{\text{cage}}[4i, 0]_{w_F}$ (along f_0) and $k_i = H_{\text{cage}}[x_i q_F, y_i q_F - 1]_{w_F}$ (just under edge f_i).

Since the coordinates $y_1, \ldots, y_n$ are all different, let σ be the permutation so that $y_{\sigma(1)} > y_{\sigma(2)} > \cdots > y_{\sigma(n)}$. It is possible to construct non-overlapping wires $Z_1, \ldots, Z_n$ of flatoms in $H_{\text{cage}} \setminus \bigcup_{i=1}^n b_i$ where wire Z_i connects flatom h_i with its bottom edge to flatom $k_{\sigma(i)}$ with its top edge, and each wire has length exactly L. This can be accomplished with a method very similar to the proof of Lemma 5, so we omit these details.

Now we can describe the unfolding of $B = B(d, (a_1, \ldots, a_n))$. Using permutation σ defined here, apply Lemma 6 to B to obtain $n+1$ wires of atoms $X_0, X_1, \ldots, X_n$. Each brick b_i will unfold to its position $(x_i q, y_i q)$ in the UCSP unfolding above. For each $1 \le i \le n$, wires X_i and Z_i were designed so that their initial edges are centered on the same unit-length segment along f_0: $e_0(X_i)[4]_1 = f_0[108i+13]_1 = e_0(Z_i)[13]_1$, and similarly their final edges are centered in the same place

on $f_{\sigma(i)}$: $e_{4L}(X_i)[4]_1 = f_{\sigma(i)}[13]_1 = e_{4L}(Z_i)[13]_1$. Furthermore, wires X_i and Z_i have the same length, $4L$, and each has an even number of left and right turns because their initial and final edges are parallel. It is thus possible, by Lemma 3, to unfold wire X_i into the region of H_{cage} described by Z_i while keeping X_i connected to both F_{floor} and $b_{\sigma(i)}$ along edges $f_0[108i + 13]_1$ and $f_{\sigma(i)}[13]_1$ respectively.

It remains to describe the unfolding of X_0, F^1_{pipe} and F^2_{pipe}. In H_{cage}, the wires $Z_1, \ldots, Z_n$ do not intersect leftmost column of modules, so define Z_0 as the wire of flatoms in $H_{\text{cage}} \cup H_{\text{drain}}$ that starts at $H_{\text{cage}}[0, 0]_{w_F}$ with its bottom edge and proceeds straight up into H_{drain} with a total length equal to the length of X_0. By Lemma 3, we may unfold X_0 into Z_0 while keeping the center of its initial edge connected to F_{floor} at $f_0[13]_1$ and the center of its final edge connected to $F^1_{\text{pipe}} \cup F^2_{\text{pipe}}$ along $g[13]_1$. In the unfolding, therefore, $F^1_{\text{pipe}} \cup F^2_{\text{pipe}}$ simply slides up relative to F_{floor} and partially juts out of the top of H_{drain}. □

5 Eliminating the Boundary

With the construction from the previous section, we are ready for the main result:

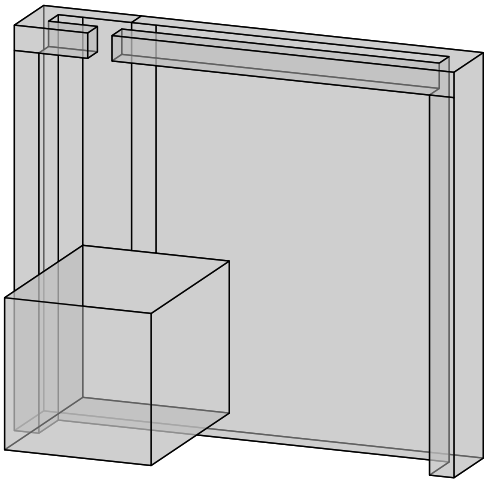
Theorem 8 *The Orthogonal Edge Unfolding problem is strongly NP-complete.*

Proof. This problem is in NP because any unfolding has integer coordinates and can thus be checked to be non-overlapping in polynomial time.

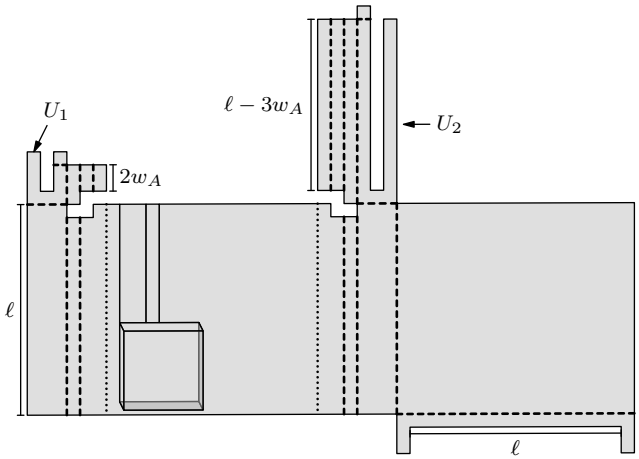
For hardness, we reduce from UCSP. For an instance $(d, (a_1, \ldots, a_n))$ of UCSP, define $B = B(d, (a_1, \ldots, a_n))$ as above, whose boundary is a square of side-length ℓ. Define the closed, orthogonal polyhedron $C = C(d, (a_1, \ldots, a_n))$ as specified in Figures 6a and 6b. The tower T (and the molecules on the tower) do not intersect the other faces of C, so this is a simple polyhedron. We will show C has an edge unfolding if and only if $(d, (a_1, \ldots, a_n))$ is a YES instance.

If $(d, (a_1, \ldots, a_n))$ is a YES instance, then by Lemma 7, there is an unfolding of B that fits inside the bounding box of F_{floor} except for $F^1_{\text{pipe}} \cup F^2_{\text{pipe}}$ which sticks above the top edge. Then Figure 6b shows that this unfolding extends to an unfolding of all of C.

On the other hand, suppose C has an edge unfolding. Let t_1 be the edge shared by F^1_{pipe} and U_1, and similarly for t_2. The shapes of U_1 and U_2 were chosen to force these two edges to be cut in the unfolding of C: indeed, if t_1 were not cut, then F^1_{pipe} and U_1 would overlap in the plane; the argument for t_2 is the same. It follows that the unfolding of C induces a connected unfolding of B, so by Lemma 4, $(d, (a_1, \ldots, a_n))$ is a YES instance. □



(a) The faces U_1 and U_2 must be cut away from B in any unfolding of C in order to avoid overlapping F_{floor}, F^1_{pipe}, or F^2_{pipe}.



(b) A partial unfolding of C showing that any unfolding of B extends to an unfolding of C.

Figure 6: The polyhedron $C = C(d, (a_1, \ldots, a_n))$ without boundary.

References

[1] M. Bern, E. D. Demaine, D. Eppstein, E. Kuo, A. Mantler, and J. Snoeyink. Ununfoldable polyhedra with convex faces. *Computational Geometry: Theory and Applications*, 24(2):51–62, February 2003.

[2] T. Biedl, E. Demaine, M. Demaine, A. Lubiw, M. Overmars, J. O'Rourke, S. Robbins, and S. Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proceedings of the 10th Canadian Conference on Computational Geometry*, Montréal, Canada, August 1998.

[3] E. D. Demaine and J. O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, July 2007.

[4] J. Y.-T. Leung, T. W. Tam, C. S. Wong, G. H. Young, and F. Y. L. Chin. Packing squares into a square. *Journal of Parallel and Distributed Computing*, 10(3):271–275, 1990.