Every Large Point Set contains Many Collinear Points or an Empty Pentagon

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Abstract

We prove the following generalised empty pentagon theorem: for every integer $\ell \geq 2$, every sufficiently large set of points in the plane contains ℓ collinear points or an empty pentagon. As an application, we settle the next open case of the "big line or big clique" conjecture of Kára, Pór, and Wood [*Discrete Comput. Geom.* 34(3):497–506, 2005].

1 Introduction

While the majority of theorems and problems about sets of points in the plane assume that the points are in general position, there are many interesting theorems and problems about sets of points with collinearities. The Sylvester-Gallai Theorem and the orchard problem are some examples; see [3]. The main contribution of this paper is to extend the 'empty pentagon' theorem about point sets in general position to point sets with collinearities.

1.1 Definitions

We begin with some standard definitions. Let P be a finite set of points in the plane. We say that P is in *general position* if no three points in P are collinear. Let $\text{conv}(P)$ denote the convex hull of P. We say that P is in *convex position* if every point of P is on the boundary of $\text{conv}(P)$. A point $v \in P$ is a *corner* of P if $\text{conv}(P - v) \neq \text{conv}(P)$. We say that P is in *strictly convex position* if each point of P is a corner of P. A *strictly convex k-gon* is the convex hull of k points in strictly convex position. If $X \subseteq P$ is a set of k points in strictly convex position and $\text{conv}(X) \cap P = X$, then $\text{conv}(X)$ is called a *k-hole* (or an *empty strictly convex k-gon*) of P. A 4-hole is called an *empty quadrilateral*, a 5-hole is called an *empty pentagon*, a 6-hole is called an *empty hexagon*, etc.

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1.2 Erdős-Szekeres Theorem

The Erdős-Szekeres Theorem [6] states that for every integer k there is a minimum integer $\text{ES}(k)$ such that every set of at least $\text{ES}(k)$ points in general position in the plane contains k points in convex position (which are therefore in strictly convex position). The following generalisation of the Erdős-Szekeres Theorem for point sets with collinearities is easily proved by applying a suitable perturbation of the points:

Theorem 1 ([1]) *For every integer k every set of at least $\text{ES}(k)$ points in the plane contains k points in convex position.*

The Erdős-Szekeres Theorem generalises for points in strictly convex position as follows:

Theorem 2 ([1]) *For all integers $\ell \geq 2$ and $k \geq 3$ there is a minimum integer $\text{ES}(k, \ell)$ such that every set of at least $\text{ES}(k, \ell)$ points in the plane contains ℓ collinear points, or k points in strictly convex position. Moreover, for all $k \geq 3$ and $\ell \geq 3$, if k is odd then*

$$\text{ES}(k, \ell) \leq \text{ES}(\tfrac{1}{2}(k-1)(\ell-1)+1) \ ,$$

and if k is even then

$$\text{ES}(k, \ell) \leq \text{ES}(\tfrac{1}{2}(k-2)(\ell-1)+2) \ .$$

Moreover, for all $k \geq 3$ and $\ell \geq 3$,

$$\text{ES}(k, \ell) \leq (\ell-3)\binom{\text{ES}(k)-1}{2} + \text{ES}(k) \ .$$

The best known upper bound on $\text{ES}(k)$, due to Tóth and Valtr [13], is

$$\text{ES}(k) \leq \binom{2k-5}{k-2} + 1 \in O\left(\frac{2^{2k}}{\sqrt{k}}\right) \ .$$

Using Theorem 2, bounds on $\text{ES}(k, \ell)$ are easily derived.

1.3 Empty Polygons

Attempting to strengthen the Erdős-Szekeres Theorem, Erdős [5] asked whether for each fixed k every sufficiently large set of points in general position contains a k-hole. Harborth [8] answered this question in the affirmative for $k \leq 5$, by showing that every set of at least ten points in general position contains a 5-hole. On the other hand, Horton [9] answered Erdős' question in the negative for $k \geq 7$, by constructing arbitrarily large sets of points in general position that contain no 7-hole. The remaining case of $k = 6$ was recently solved independently by Gerken [7] and Nicolás [12], who proved that every sufficiently large set of points in general position contains a 6-hole. The above results do not easily generalise to sets with a bounded number of collinear points (as in Theorem 2). Nevertheless, we prove the following 'generalised empty pentagon' theorem, which is the main contribution of this paper.

Theorem 3 *For every integer $\ell \geq 2$, every finite set of at least $\mathrm{ES}(\frac{(2\ell-1)^\ell-1}{2\ell-2})$ points in the plane contains ℓ collinear points, or a 5-hole.*

Note that Eppstein [4] characterised the point sets with no 5-hole in terms of the acyclicity of an associated *quadrilateral graph*. However, it is not clear how this result can be used to prove Theorem 3.

1.4 Big Line or Big Clique Conjecture

Theorem 3 has an important ramification for the following "big line or big clique" conjecture by Kára et al. [10]. Let P be a finite set of points in the plane. Distinct points $v, w \in P$ are *visible* with respect to P if $P \cap \overline{vw} = \{v, w\}$, where $\overline{vw}$ denotes the closed line segment between v and w. The *visibility graph* of P has vertex set P, where distinct points $v, w \in P$ are adjacent if and only if they are visible with respect to P.

Conjecture 1 ([10]) *For all integers k and ℓ there is an integer n such that every finite set of at least n points in the plane contains ℓ collinear points, or k pairwise visible points (that is, the visibility graph contains a k-clique).*

Conjecture 1 has recently attracted considerable attention [2, 10, 11]. It is trivially true for $\ell \leq 3$ and all k. Kára et al. [10] proved it for $k \leq 4$ and all ℓ. Addario-Berry et al. [2] proved it in the case that $k = 5$ and $\ell = 4$. Here we prove the next case of Conjecture 1 for infinitely many values of ℓ.

Theorem 4 *Conjecture 1 is true for $k = 5$ and all ℓ.*

Proof. By Theorem 3, every sufficiently large set of points contains ℓ collinear points (in which case we are done) or a 5-hole H. Let H' be a 5-hole contained in H with minimum area. Then the corners of H' are five pairwise visible points (otherwise there is a 5-hole contained in H with less area; see Figure 1). $\square$

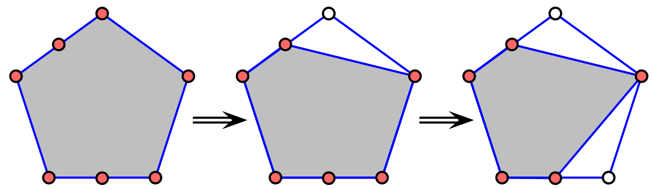


Figure 1: Every 5-hole contains 5 pairwise visible points.

2 Proof of Theorem 3

The proof of Theorem 3 loosely follows the proof of the 6-hole theorem for points in general position by Valtr [14], which in turn is a simplification of the proof by Gerken [7]. For distinct points a, b, c in the plane, let $\Delta[a, b, c]$ be the closed triangle determined by a, b, c, and let $\Delta(a, b, c)$ be the open triangle determined by a, b, c. For integers $n \leq m$, let $[n, m] := \{n, n+1, \ldots, m\}$ and $[n] := [1, n]$.

Consider the following problem (which is also relevant to the proof of Theorem 2): given a set P of points in convex position, choose a large subset of P in strictly convex position. For integers $k \geq 1$ and $\ell \geq 1$, let $q(k, \ell)$ be the minimum integer such that every set of at least $q(k, \ell)$ points in the plane in convex position contains ℓ collinear points or k points in strictly convex position. It is reasonably straightforward to determine $q(k, \ell)$ as follows [1]:

$$q(k,\ell) = \begin{cases} \min\{k,\ell\} & \text{if } k \leq 2 \text{ or } \ell \leq 2 \\ \ell & \text{for } \ell \geq 1 \text{ and } k = 3 \\ k & \text{for } \ell = 3 \text{ and } k \geq 1 \\ \frac{1}{2}(\ell-1)(k-1)+1 & \text{for } \ell \geq 3 \text{ and odd } k \geq 3 \\ \frac{1}{2}(\ell-1)(k-2)+2 & \text{for } \ell \geq 3 \text{ and even } k \geq 4 \end{cases}$$

Proof of Theorem 3. Fix $\ell \geq 3$ and let $k := \frac{(2\ell-1)^\ell-1}{2\ell-2}$, which is an integer. Let P be a set of at least $\mathrm{ES}(k)$ points in the plane. By Theorem 1, P contains k points in convex position. Suppose for the sake of contradiction that P contains no ℓ collinear points and no 5-hole.

A set X of at least k points in P in convex position is said to be *k-minimal* if there is no set Y of at least k points in P in convex position, such that $\mathrm{conv}(Y) \subsetneq \mathrm{conv}(X)$. As illustrated in Figure 2, let A_1 be a k-minimal subset of P. Let $A_2, \ldots, A_{\ell-1}$ be the convex layers inside A_1. More precisely, for $i = 2, \ldots, \ell-1$, let A_i be the set of points in P on the boundary of

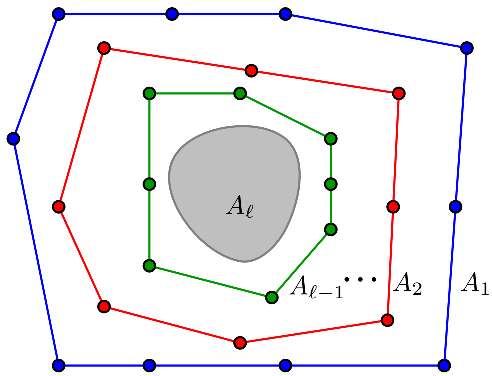


Figure 2: Definition of $A_1, \dots, A_\ell$.

the convex hull of $(P \cap \text{conv}(A_{i-1})) - A_{i-1}$. Let $A_\ell := (P \cap \text{conv}(A_{\ell-1})) - A_{\ell-1}$.

Since $q(5, \ell) = 2\ell - 1$, for each $i \in [2, \ell]$, any $2\ell - 1$ consecutive points of A_{i-1} contains five points in strictly convex position. Thus the convex hull of any $2\ell - 1$ consecutive points of A_{i-1} contains a point in A_i, as otherwise it would contain a 5-hole. Now A_{i-1} contains $\left\lfloor \frac{|A_{i-1}|}{2\ell-1} \right\rfloor$ disjoint subsets, each consisting of $2\ell - 1$ consecutive points, and the convex hull of each subset contains a point in A_i. Since the convex hulls of these subsets of A_{i-1} are disjoint,

$$|A_i| \geq \left\lfloor \frac{|A_{i-1}|}{2\ell - 1} \right\rfloor > \frac{|A_{i-1}|}{2\ell - 1} - 1 \ ,$$

implying $|A_{i-1}| < (2\ell-1)(|A_i|+1)$. Suppose that $A_i = \varnothing$ for some $i \in [2, \ell]$. Thus $|A_{i-1}| < 2\ell - 1$ and $|A_{i-2}| < (2\ell - 1)^2 + (2\ell - 1)$, and by induction,

$$|A_1| < \sum_{j=1}^{i-1} (2\ell-1)^j < \frac{(2\ell - 1)^i - 1}{2\ell - 2} \leq \frac{(2\ell - 1)^\ell - 1}{2\ell - 2} = k \ ,$$

which is a contradiction. Now assume that $A_i \neq \varnothing$ for all $i \in [\ell]$. Fix a point $z \in A_\ell$.

Note that if $|A_i| \leq 2$ for some $i \in [\ell - 1]$ then $A_{i+1} = \varnothing$. Thus we may assume that $|A_i| \geq 3$ for all $i \in [\ell - 1]$. Consider each such set A_i to be ordered clockwise around $\text{conv}(A_i)$. If x and y are consecutive points in A_i with y clockwise from x then we say that the oriented segment $\overrightarrow{xy}$ is an *arc* of A_i.

Let $\overrightarrow{xy}$ be an arc of A_i for some $i \in [\ell - 2]$. We say that $\overrightarrow{xy}$ is *empty* if $\Delta(x, y, z) \cap A_{i+1} = \varnothing$, as illustrated in Figure 3(a). In this case, the intersection of the boundary of $\text{conv}(A_{i+1})$ and $\Delta(x, y, z)$ is contained in an arc $\overrightarrow{pq}$. We call $\overrightarrow{pq}$ the *follower* of $\overrightarrow{xy}$.

Claim 1 *If $\overrightarrow{pq}$ is the follower of an empty arc $\overrightarrow{xy}$, then $\{x, y, p, q\}$ is a 4-hole and $\overrightarrow{pq}$ is empty.*

Proof. Say $\overrightarrow{xy}$ is an arc of A_i, where $i \in [\ell - 2]$. Let $S := \{x, y, p, q\}$. Since p and q are in the interior of

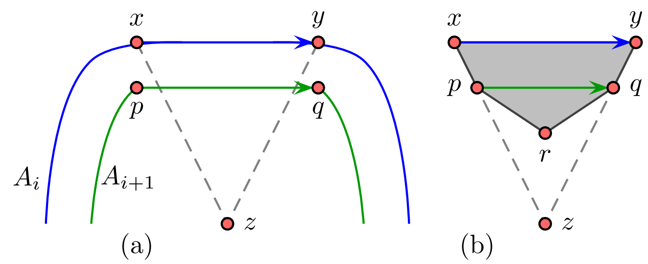


Figure 3:

$\text{conv}(A_i)$, both x and y are corners of S. Both p and q are corners of S, as otherwise $\overrightarrow{xy}$ is not empty. Thus S is in strictly convex position. S is empty by the definition of A_{i+1}. Thus S is a 4-hole.

Suppose that $\overrightarrow{pq}$ is not empty; that is, $\Delta(p, q, z) \cap A_{i+2} \neq \varnothing$. Let r be a point in $\Delta(p, q, z) \cap A_{i+2}$ closest to $\overline{pq}$. Thus $\Delta(p, q, r) \cap P = \varnothing$. Since $\{x, y, p, q\}$ is a 4-hole, $\{x, y, p, q, r\}$ is a 5-hole, as illustrated in Figure 3(b). This contradiction proves that $\overrightarrow{pq}$ is empty. $\square$

As illustrated in Figure 4(a)–(c), we say the follower $\overrightarrow{pq}$ of $\overrightarrow{xy}$ is:

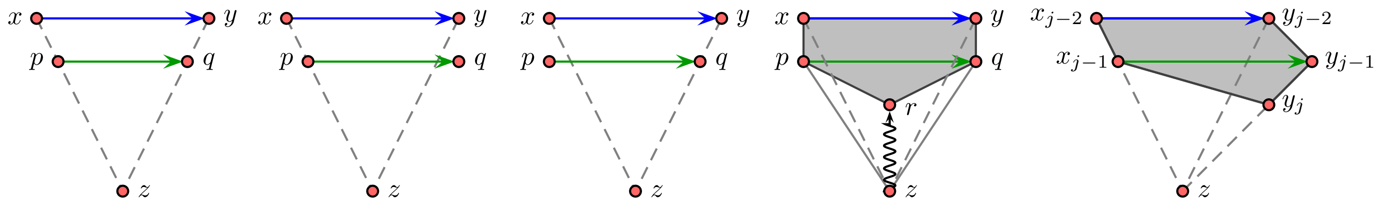
- *double-aligned* if $p \in \overline{xz}$ and $q \in \overline{yz}$,
- *left-aligned* if $p \in \overline{xz}$ and $q \notin \overline{yz}$,
- *right-aligned* if $p \notin \overline{xz}$ and $q \in \overline{yz}$.

Claim 2 *If $\overrightarrow{pq}$ is the follower of an empty arc $\overrightarrow{xy}$, then $\overrightarrow{pq}$ is either double-aligned or left-aligned or right-aligned.*

Proof. Suppose that $\overrightarrow{pq}$ is neither double-aligned nor left-aligned nor right-aligned, as illustrated in Figure 4(d). Since $\overrightarrow{xy}$ is empty, $p \notin \Delta[x, y, z]$ and $q \notin \Delta[x, y, z]$. Let $D := (P \cap \Delta[p, q, z]) - \{p, q\}$. Thus $z \in D$ and $D \neq \varnothing$. Let r be a point in D closest to $\overline{pq}$. Thus $\Delta(r, p, q)$ is empty. By Claim 1, $\{x, y, p, q\}$ is a 4-hole. Thus $\{x, y, p, q, r\}$ is a 5-hole, which is the desired contradiction. $\square$

Suppose that no arc of A_1 is empty. That is, $\Delta(x, y, z) \cap A_2 \neq \varnothing$ for each arc $\overrightarrow{xy}$ of A_1. Observe that $\Delta(x, y, z) \cap \Delta(p, q, z) = \varnothing$ for distinct arcs $\overrightarrow{xy}$ and $\overrightarrow{pq}$ of A_1 (since these triangles are open). Thus $|A_2| \geq |A_1|$, which contradicts the minimality of A_1.

Now assume that some arc $\overrightarrow{x_1y_1}$ of A_1 is empty. For $i = 2, 3, \dots, \ell - 1$, let $\overrightarrow{x_iy_i}$ be the follower of $\overrightarrow{x_{i-1}y_{i-1}}$. By Claim 1 (at each iteration), $\overrightarrow{x_iy_i}$ is empty. For some $i \in [2, \ell - 2]$, the arc $\overrightarrow{x_iy_i}$ is not double-aligned, as otherwise $\{x_1, x_2, \dots, x_{\ell-2}, z\}$ are collinear and $\{y_1, y_2, \dots, y_{\ell-2}, z\}$ are collinear, which implies that $\{x_1, x_2, \dots, x_{\ell-1}, z\}$ are collinear or $\{y_1, y_2, \dots, y_{\ell-1}, z\}$ are collinear by Claim 2. Let i be the minimum integer in $[2, \ell-2]$ such that $\overrightarrow{x_iy_i}$ is not double-aligned. Without loss of generality, $\overrightarrow{x_iy_i}$ is left-aligned. On the other hand,



(a) double-aligned (b) left-aligned (c) right-aligned (d) neither (e)

Figure 4:

$\overrightarrow{x_j y_j}$ is not left-aligned for all $j \in [i+1, \ell-1]$, as otherwise $\{x_1, x_2, \dots, x_{\ell-1}, z\}$ are collinear. Let j be the minimum integer in $[i+1, \ell-1]$ such that $\overrightarrow{x_j y_j}$ is not left-aligned. Thus $\overrightarrow{x_{j-1} y_{j-1}}$ is left-aligned and $\overrightarrow{x_j y_j}$ is not left-aligned. It follows that $\{x_{j-2}, y_{j-2}, y_{j-1}, y_j, x_{j-1}\}$ is a 5-hole, as illustrated in Figure 4(e). This contradiction proves that P contains ℓ collinear points or a 5-hole. □

We expect that the lower bound on $|P|$ in Theorem 3 is far from optimal. All known point sets with at most ℓ collinear points and no 5-hole have $O(\ell^2)$ points, the $\ell \times \ell$ grid for example. See [4, 10] for other examples.

Open Problem. For which values of ℓ is there an integer n such that every set of at least n points in the plane contains ℓ collinear points or a 6-hole?

This is true for $\ell = 3$ by the empty hexagon theorem. If this question is true for a particular value of ℓ then Conjecture 1 is true for $k = 6$ and the same value of ℓ. For $k \geq 7$ different methods are needed since there are point sets in general position with no 7-hole.

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