Folding Equilateral Plane Graphs

Zachary Abel[1] Erik D. Demaine[2] Martin L. Demaine[2] Sarah Eisenstat[2] Jayson Lynch[2] Tao B. Schardl[2] Isaac Shapiro-Ellowitz[3]

[1]MIT Department of Mathematics

[2]MIT CSAIL

[3]University of Massachusetts Boston

ISAAC 2011, Yokohama, Japan

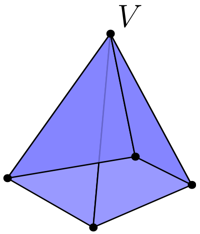
Recurring Notions

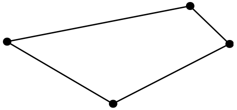
- Weighted plane graph: a graph $G = (V, E)$ with edge lengths $\ell : E \to \mathbb{R}_{>0}$ and a preferred *combinatorial* planar embedding
  - Parallel edges OK, no self loops
  - All graphs are weighted unless otherwise specified
- Self-touching linkage [CDR02]: A straight-line drawing of a weighted graph that does not properly cross itself (but may touch itself)
- A linkage realizing a weighted plane graph G is a self-touching configuration of G.
- Linear configuration: geometrically collinear

Questions from Origami Motivation

Does a given weighted plane graph have a linear configuration?
(Can it be instantaneously "folded flat"?)

- "1-D Origami"
- Single-vertex origami:
  - Can a creased, 1-vertex cone be folded flat?
  - Our question for a cycle where edge lengths = angles at vertex
  - INTEGER PARTITION $\implies$ weakly NP-Hard

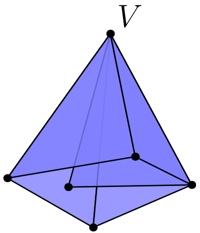


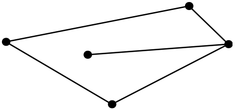


Questions from Origami Motivation

Does a given weighted plane graph have a linear configuration?
(Can it be instantaneously "folded flat"?)

- "1-D Origami"
- Single-vertex origami:
  - Can a creased, 1-vertex cone be folded flat?
  - Our question for a cycle where edge lengths = angles at vertex
  - Integer Partition $\Longrightarrow$ weakly NP-Hard
- Single-vertex polyhedral complex folding:
  - Exactly our problem (if "outside region" specified)

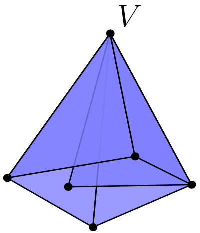


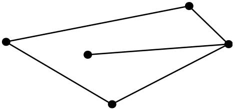


Questions from Origami Motivation

Does a given weighted plane graph have a linear configuration?
(Can it be instantaneously "folded flat"?)

- "1-D Origami"
- Single-vertex origami:
  - Can a creased, 1-vertex cone be folded flat?
  - Our question for a cycle where edge lengths = angles at vertex
  - INTEGER PARTITION $\Longrightarrow$ weakly NP-Hard
- Single-vertex polyhedral complex folding:
  - Exactly our problem (if "outside region" specified)
- Results:
  - General graphs: **strongly NP-hard**
  - Equilateral: **flat-foldable** $\Longleftrightarrow$ **bipartite**

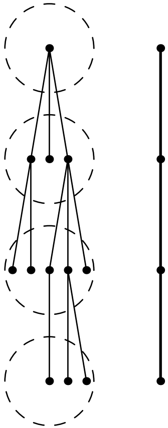




Questions from Linkage Folding Motivation

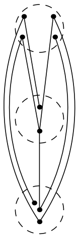
- Canonical tree configuration: all edges move down away from root
  - This is linear
  - All canonical configurations are connected by continuous motion.

Can all configured trees be continuously canonicalized (“unfolded”)?

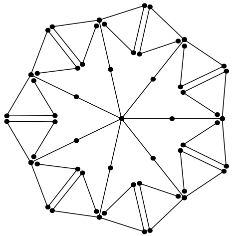


Questions from Linkage Folding Motivation

- Canonical tree configuration: all edges move down away from root
  - This is linear
  - All canonical configurations are connected by continuous motion.



Can all configured trees be continuously canonicalized ("unfolded")?

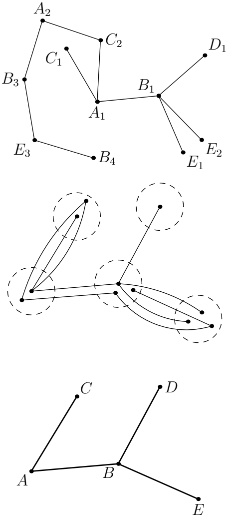


- Previously known [BCD+09]:
  - Linear trees? No
  - Equilateral trees? No
- Result:
  - Linear *and* equilateral? **Yes!**

Plane Graph Homomorphisms

- All graphs equilateral
- A plane graph homomorphism $f : G \to H$ is
  - a graph homomorphism $f : G \to H$ and
  - for each edge $e \in E(H)$ a chosen ordering for edges $f^{-1}(e)$ to stack along e

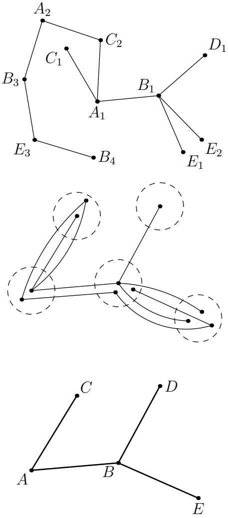
  satisfying planarity and respect for G's embedding.
  - A "self-touching configuration of G along H"



Plane Graph Homomorphisms

- All graphs equilateral
- A plane graph homomorphism $f : G \to H$ is
  - a graph homomorphism $f : G \to H$ and
  - for each edge $e \in E(H)$ a chosen ordering for edges $f^{-1}(e)$ to stack along e

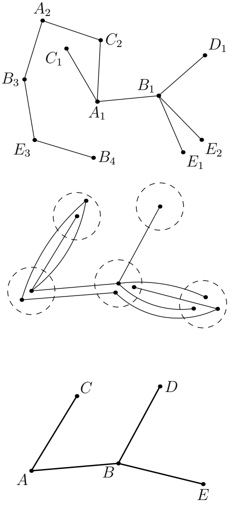
  satisfying planarity and respect for G's embedding.
  - A "self-touching configuration of G along H"
- A configuration of H induces a configuration of G via $G \to H$
  - Continuous motions are induced similarly



Plane Graph Homomorphisms

- All graphs equilateral
- A plane graph homomorphism $f : G \rightarrow H$ is
  - a graph homomorphism $f : G \rightarrow H$ and
  - for each edge $e \in E(H)$ a chosen ordering for edges $f^{-1}(e)$ to stack along e

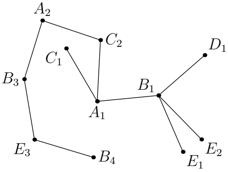
  satisfying planarity and respect for G's embedding.
  - A "self-touching configuration of G along H"
- A configuration of H induces a configuration of G via $G \rightarrow H$
  - Continuous motions are induced similarly

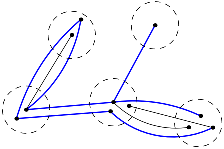


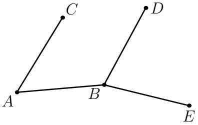
Plane Graph Homomorphisms

- All graphs equilateral
- A plane graph homomorphism $f : G \to H$ is
  - a graph homomorphism $f : G \to H$ and
  - for each edge $e \in E(H)$ a chosen ordering for edges $f^{-1}(e)$ to stack along e

  satisfying planarity and respect for G's embedding.
  - A "self-touching configuration of G along H"
- A configuration of H induces a configuration of G via $G \to H$
  - Continuous motions are induced similarly
- Boundary edges of $f : G \to H$







Unfolding Linear Equilateral Trees

Theorem

Any linear configuration of an equilateral tree G can be continuously deformed into a canonical configuration. So all linear configurations are continuously connected.

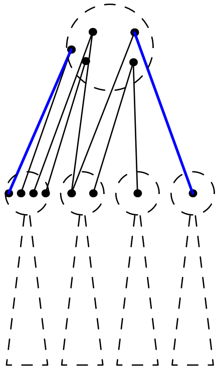
- *k-canonical*: a configuration induced from surjective $G \twoheadrightarrow H_k$, where H_k is a canonical k-vertex tree
  - An n-canonical configuration is canonical (where $n = |V(G)|$)
  - Initially $G \twoheadrightarrow P_r$ ($2 \leq r \leq n$), so r-canonical

Unfolding Linear Equilateral Trees

Theorem

Any linear configuration of an equilateral tree G can be continuously deformed into a canonical configuration. So all linear configurations are continuously connected.

- *k*-canonical: a configuration induced from surjective $G \twoheadrightarrow H_k$, where H_k is a canonical k-vertex tree
  - An n-canonical configuration is canonical (where $n = |V(G)|$)
  - Initially $G \twoheadrightarrow P_r$ $(2 \leq r \leq n)$, so r-canonical
- Move from k-canonical to $(k+1)$-canonical
  - Find a break in the boundary edges.
  - Move break to root
  - Split root: $G \twoheadrightarrow H_{k+1} \twoheadrightarrow H_k$
  - Canonicalize H_{k+1}

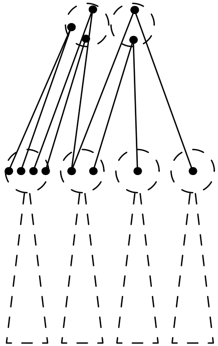


Unfolding Linear Equilateral Trees

Theorem

Any linear configuration of an equilateral tree G can be continuously deformed into a canonical configuration. So all linear configurations are continuously connected.

- *k-canonical*: a configuration induced from surjective $G \twoheadrightarrow H_k$, where H_k is a canonical k-vertex tree
  - An n-canonical configuration is canonical (where $n = |V(G)|$)
  - Initially $G \twoheadrightarrow P_r$ ($2 \leq r \leq n$), so r-canonical
- Move from k-canonical to $(k+1)$-canonical
  - Find a break in the boundary edges.
  - Move break to root
  - Split root: $G \twoheadrightarrow H_{k+1} \twoheadrightarrow H_k$
  - Canonicalize H_{k+1}

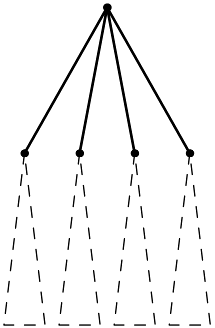


Unfolding Linear Equilateral Trees

Theorem

Any linear configuration of an equilateral tree G can be continuously deformed into a canonical configuration. So all linear configurations are continuously connected.

- k-canonical: a configuration induced from surjective $G \twoheadrightarrow H_k$, where H_k is a canonical k-vertex tree
  - An n-canonical configuration is canonical (where $n = |V(G)|$)
  - Initially $G \twoheadrightarrow P_r$ $(2 \le r \le n)$, so r-canonical
- Move from k-canonical to $(k+1)$-canonical
  - Find a break in the boundary edges.
  - Move break to root
  - Split root: $G \twoheadrightarrow H_{k+1} \twoheadrightarrow H_k$
  - Canonicalize H_{k+1}

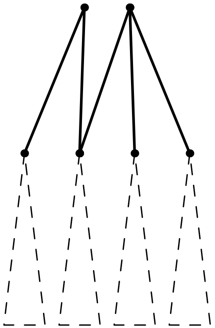


Unfolding Linear Equilateral Trees

Theorem

Any linear configuration of an equilateral tree G can be continuously deformed into a canonical configuration. So all linear configurations are continuously connected.

- *k*-canonical: a configuration induced from surjective $G \twoheadrightarrow H_k$, where H_k is a canonical k-vertex tree
  - An n-canonical configuration is canonical (where $n = |V(G)|$)
  - Initially $G \twoheadrightarrow P_r$ $(2 \leq r \leq n)$, so r-canonical
- Move from k-canonical to $(k+1)$-canonical
  - Find a break in the boundary edges.
  - Move break to root
  - Split root: $G \twoheadrightarrow H_{k+1} \twoheadrightarrow H_k$
  - Canonicalize H_{k+1}

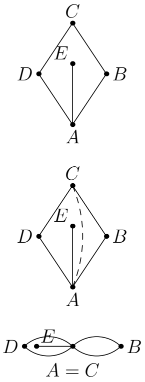


Folding Equilateral Graphs

Theorem

An equilateral graph G has a linear configuration if and only if it is bipartite.

- Linear $\implies$ bipartite:
  - $G \twoheadrightarrow P_r$ and P_r bipartite $\implies$ G bipartite
- Bipartite $\implies$ linear: induct on $n = |V(G)|$
  - If $n = 2$, it is linear
  - Else, some face $F = v_1 v_2 \cdots v_r$ has $r \geq 3$
  - WLOG, v_1, v_2, v_3 distinct
  - Add edge $e = (v_1, v_3)$ inside F and contract e
  - gives $G \rightarrow H$, where H is smaller and bipartite

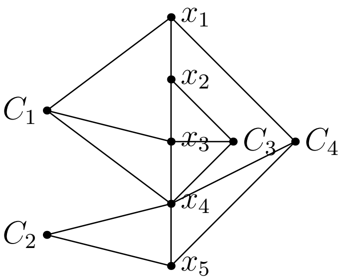


Folding General Graphs

Theorem

The problem of deciding if a weighted plane graph has a linear configuration is strongly NP-hard.

- Reduce from Planar Monotone 3-Sat [dBK10]:
  - Variables in vertical column
  - Positive 3-Sat clauses on right, negative on left
- Variable gadget:
  - Duplicate variables
  - (left, right) = (true, false)
- Clause Gadget: probe, blockers

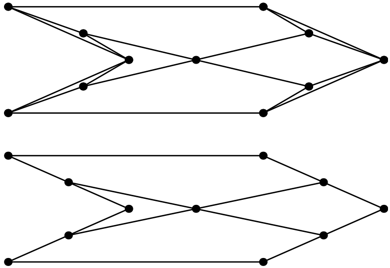


Folding General Graphs

Theorem

The problem of deciding if a weighted plane graph has a linear configuration is strongly NP-hard.

- Reduce from PLANAR MONOTONE 3-SAT [dBK10]:
  - Variables in vertical column
  - Positive 3-Sat clauses on right, negative on left
- Variable gadget:
  - Duplicate variables
  - (left, right) = (true, false)
- Clause Gadget: probe, blockers

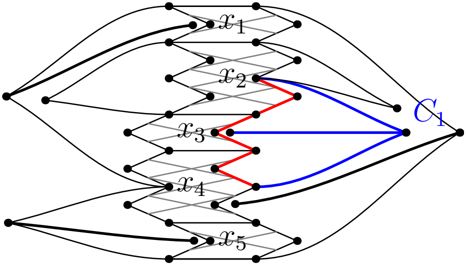


Folding General Graphs

Theorem

The problem of deciding if a weighted plane graph has a linear configuration is strongly NP-hard.

- Reduce from PLANAR MONOTONE 3-SAT [dBK10]:
  - Variables in vertical column
  - Positive 3-Sat clauses on right, negative on left
- Variable gadget:
  - Duplicate variables
  - (left, right) = (true, false)
- Clause Gadget: probe, blockers



Open Questions

- Detect when two configurations of a graph are connected by motion.
  - Relationship with existential theory of reals?
  - Special cases? E.g., determine if a configured tree can be canonicalized.
- Natural weighted graph families where instantaneous flat folding problem is easy?
- Origami with polygonal complexes