COMBINATORIAL PROPERTIES OF THE HIGMAN-SIMS GRAPH

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1. Introduction

In this survey we discuss properties of the Higman-Sims graph, which has 100 vertices, 1100 edges, and is 22 regular. In fact it is *strongly* regular, where any two adjacent nodes have 0 common neighbors, and any two non-adjacent nodes have 6 common neighbors. We offer multiple constructions of the graph, based on the unique $S(3, 6, 22)$ Steiner system (obtained by extending the projective plain Π_4), the Leech lattice, and maximal cocliques in the $(50, 7, 0, 1)$ strongly regular Hoffman-Singleton graph. Along the way, we compute the automorphism group of the Higman-Sims graph, proving that it is a double cover of a simple group, the so-called Higman-Sims group.

2. Construction and Uniqueness

In this section we construct the Higman-Sims graph G_{HS} and prove that it is the unique $(100, 22, 0, 6)$ strongly regular graph. Both the construction and the proof of uniqueness rely heavily on the $S(3, 6, 22)$ Steiner system, D_{22}, which is known to be unique for those parameters. We require no knowledge about D_{22} other than the fact that it is the unique 3-$(22, 6, 1)$ design, and we will make frequent use of its intersection triangle shown in Figure 1, where entry $\nu_{i,j}$ for $0 \le j \le i \le 6$ counts the number of blocks of D_{22} that intersect a given set of i points of a block exactly in a prescribed subset of size j.

Let $D_{22} = (X, \mathcal{B})$. Define a graph G_{HS} on vertex set $V(G_{\mathrm{HS}}) = \{\star\} \cup X \cup \mathcal{B}$ of size $1 + 22 + 77 = 100$, with edges defined as follows:

- The point $\star$ connects to each $x \in X$;
- for $x \in X$ and $B \in \mathcal{B}$, edge (x, B) is in the graph if and only if $x \in B$;
- and for $B_1, B_2 \in \mathcal{B}$, edge (B_1, B_2) is in the graph if and only if $B_1 \cap B_2 = \emptyset$.

Theorem 1. *The graph G_{HS} defined above is strongly regular with parameters* $(100, 22, 0, 6)$.

Proof. In what follows, x and x_i are always elements of X, and B or B_i are elements of $\mathcal{B}$.

To see that G_{HS} is regular of degree 22, note that $\star$ connects to the 22 elements of X; each $x \in X$ connects to $\star$ and the 21 blocks containing it; and each block $B \in \mathcal{B}$ connects to its 6 points and the 16 blocks disjoint from it.

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$$\begin{array}{ccccccccccccc}
 & & & & & & 77 & & & & & & \\
 & & & & & 56 & & 21 & & & & & \\
 & & & & 40 & & 16 & & 5 & & & & \\
 & & & 28 & & 12 & & 4 & & 1 & & & \\
 & & 20 & & 8 & & 4 & & 0 & & 1 & & \\
 & 16 & & 4 & & 4 & & 0 & & 0 & & 1 & \\
16 & & 0 & & 4 & & 0 & & 0 & & 0 & & 1
\end{array}$$

Figure 1: The intersection triange for the $S(3,6,22)$ Steiner system D_{22}. Entry $\nu_{i,j}$ counts the number of blocks of D_{22} that intersect a given set of i points from a block exactly in a given subset of size j.

The property $\lambda = 0$ is equivalent to G_{HS} having no triangles. There can be no triangle of the form x, B_1, B_2 because if $x \in B_1$ and $x \in B_2$ then B_1 and B_2 are not disjoint. The only other possibility is a triangle of blocks B_1, B_2, B_3, so let us show that no such triangle exists. Suppose that B_1 and B_2 are disjoint blocks. How many blocks B intersect B_1 and B_2 (necessarily in two points each)? Such a block B may be specified by choosing the two points $\{x_1, x_2\} = B_1 \cap B$ and a point $x_3 \in B_2 \cap B$, and every block B comes from two such choices (we could replace x_3 with the other point of $B \cap B_2$), so there are $\binom{6}{2} \cdot 6/2 = 45$ blocks intersecting both B_1 and B_2. It follows that there are $75 - 60 - 60 + 45 = 0$ blocks disjoint from B_1 and B_2, as needed.

Now we verify $\mu = 6$. A pair of non-adjacent edges has the form $(\star, B)$, (x_1, x_2), (x, B) with $x \notin B$, or (B_1, B_2) with $B_1 \cap B_2 \neq \emptyset$. In the first case, $\star$ and B have six common neighbors, namely the elements of B. In the (x_1, x_2) case, both points are connected to $\star$ and are contained in exactly 5 blocks of D_{22}. Next suppose $x \notin B$. Any block $B' \neq B$ intersects B in 0 or 2 points, and for each pair of points $x_1, x_2 \in B$ there is a unique block containing x, x_1, x_2, so there are exactly $\binom{6}{2}$ blocks containing x that intersect B. So there are $21 - 15 = 6$ blocks containing x that are disjoint from B, *i.e.* nodes $x, B \in V(G_{\text{HS}})$ have exactly 6 common neighbors. Finally, suppose blocks B_1, B_2 are not disjoint, and therefore intersect in 2 points, thus accounting for two common neighbors of B_1 and B_2. With a similar count as in the previous paragraph, we find that there are exactly 4 blocks disjoint from both B_1 and B_2, showing that B_1 and B_2 indeed have 6 common neighbors. This completes the proof of strong regularity. $\square$

We may also use the uniqueness of D_{22} to show the uniqueness of the G_{HS} graph. This uniqueness was first proved by Gewirtz [3], but the proof below follows a simplified argument by Brouwer [1].

Theorem 2. *The Higman-Sims graph* G_{HS} *is the unique* $(100, 22, 0, 6)$ *strongly regular graph.*

Proof. Suppose G is some $(100, 22, 0, 6)$ graph. Pick any vertex $\star' \in V(G)$, and let Y be the set of 22 neighbors of $\star'$, with $\mathcal{C}$ the set of 77 non-neighbors of $\star'$. Any node $C \in \mathcal{C}$ has $\mu = 6$ neighbors in common with $\star'$, so let $N(C)$ denote this set of six points in Y that are neighbors of C. The crux of the proof is showing that these 77 6-tuples $N(C)$ form a 3-$(22, 6, 1)$ design on Y.

First, if two point $C_1, C_2 \in \mathcal{C}$ had $|N(C_1) \cap N(C_2)| \geq 3$, then these three points of Y together with $C_1, C_2, \star'$ would form a complete bipartite induced subgraph $K \cong K_{3,3}$. But the variance trick allows us to prove that no such subgraph exists. Indeed, for each $0 \leq i \leq 6$ let n_i denote the number of nodes outside of K adjacent to exactly i nodes of K. The number of nodes in $G \setminus K$ is $\sum_i n_i = 94$. Each node of K connects to 19 nodes outside of K, so the number of edges joining K and $G \setminus K$ is $6 \cdot 19 = \sum_i i n_i$. Finally, for each pair of distinct nodes $\{k_1, k_2\}$ in K, if they are not adjacent then they have 3 common neighbors in K and $\mu - 3 = 3$ common neighbors outside K, whereas if they are adjacent then they have no common neighbors. So the number of unordered paths (k_1, v, k_2) in G with $k_1 \neq k_2 \in K$ and $v \in G \setminus K$ is $6 \cdot 3 = \sum_i \binom{i}{2} n_i$. It follows that $\sum_i (i-1)(i-2)n_i = -2$, which is impossible because $(i-1)(i-2)n_i \geq 0$ for each i. So no induced $K_{3,3}$ subgraphs exist in G, meaning any two blocks $N(C_1)$ and $N(C_2)$ have at most 2 points in common.

We have just shown that each triple of points in Y is contained in at most one block $N(C)$, so exactly $\binom{6}{3} \cdot 77 = 1540$ distinct triples in Y appear in some block. But there are exactly $\binom{22}{3} = 1540$ triples of points in Y, so each appears exactly once. This exactly means that the sets $N(C)$ indeed form a $S(3, 6, 22)$ Steiner system on Y, which must be the unique D_{22} design.

If two points $C_1, C_2 \in \mathcal{C}$ are adjacent in G then they have no common neighbors, so $N(C_1) \cap N(C_2) = \emptyset$. It follows that the 16 neighbors of C_1 in $\mathcal{C}$ must correspond exactly to the 16 blocks in the $S(3, 6, 22)$ Steiner system disjoint from $N(C_1)$. So G has exactly the same edges as G_{HS} above, as desired. □

Note. The graph whose 77 vertices are the blocks of D_{22} where two blocks are adjacent if and only if they are disjoint—equivalently, the subgraph of G_{HS} induced on the vertices $\mathcal{B}$—forms a strongly regular graph with parameters $(77, 16, 0, 4)$. Brouwer [1] uses this embedding into the Higman-Sims graph, combined with methods similar to those above, to show the uniqueness of the 77-node strongly regular graph with the above parameters.

3. Automorphisms of G_{HS} and the Higman-Sims group

Let $\overline{HS} = \mathrm{Aut}(G_{\mathrm{HS}})$ be the automorphism group of graph G_{HS}. An automorphism $f \in \mathrm{Aut}(G_{\mathrm{HS}})$ fixing $\star$ must send X to itself and $\mathcal{B}$ to itself while preserving incidence, *i.e.* must be an automorphism of D_{22}. So the point stabilizer $\overline{HS}_\star$ is $\mathrm{Aut}(D_{22}) = \overline{M}_{22}$, which contains the Mathieu group M_{22} with index 2 as the subgroup of even permutations of the 99 elements $X \cup \mathcal{B}$. The group $\overline{HS}$, thought of as a permutation group on its 100 nodes, therefore contains odd permutations, so we may define $HS \subset \overline{HS}$ as the index-2 subgroup of even permutations, whose point stabilizer $HS_\star$ is M_{22}. In this section we will compute the orders of $\overline{HS}$ and HS and show that the group HS is simple.

Lemma 3. *The group HS acts transitively on the vertices of G_{HS}.*

Proof. Recall that M_{22} acts three-transitively on the points of D_{22}, so $HS_\star$ acts transitively on $X \subset G_{\text{HS}}$ and acts transitively on $\mathcal{B} \subset G_{\text{HS}}$ (because three points uniquely determine their block). Let $x \in X$ be any point, and let Y and $\mathcal{C}$ be the set of neighbors and non-neighbors of x, respectively. By the uniqueness of the Higman-Sims graph, the neighbor sets $\{y \in Y \mid (C, y) \in E(G_{\text{HS}})\}$ for each $C \in \mathcal{C}$ form a D_{22} on Y. Thus, the point stabilizer HS_y is another copy of M_{22}, which therefore acts transitively on Y. But Y contains $\star$ as well as points from X and $\mathcal{B}$, so all 100 nodes of G_{HS} are indeed in the same orbit under HS. □

Corollary 4. *The group HS has order $100 \cdot |M_{22}| = 44352000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, and $\overline{HS}$ has order $2 \cdot |HS| = 88704000 = 2^{10} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$.*

Proof. This follows directly from the Orbit-Stabilizer theorem. □

Corollary 5. *The group HS acts primitively on the vertices of G_{HS}.*

Proof. If G_{HS} were transitive but imprimitive, then its action would fix some partition $V(G_{\text{HS}}) = P_1 \cup \cdots \cup P_k$ for $2 < k < 100$. Because G_{HS} acts transitively, all P_i have the same size d, which is some divisor of 100. Suppose without loss of generality that $\star \in P_1$. Because neither $1 + 22$ nor $1 + 77$ divide 100, we may find two vertices $v, w \in V(G_{\text{HS}}) \setminus \{\star\}$ such that $v \in P_1$, $w \notin P_1$, and either both v and w are in X or both are in $\mathcal{B}$. We know that $HS_\star = M_{22}$ acts transitively on both X and $\mathcal{B}$, so there is some automorphism of G_{HS} fixing $\star$ and sending v to w. This automorphism does not preserve the partition $P_1 \cup \cdots \cup P_k$, contrary to assumption. □

Armed with the primitivity of the action of HS on $V(G_{\text{HS}})$, as well as the fact that the point stabilizers of HS—namely M_{22}—are simple, we may prove the simplicity of HS itself:

Theorem 6. *The group HS is simple.*

Proof. Suppose $\{1\} \subsetneq N \subsetneq HS$ is a normal subgroup. If N were not transitive, then the orbits under the action of N would form a partition of $V(G_{\text{HS}})$ which is fixed by the action of HS, contrary to the fact that HS acts primitively. So N is transitive. The intersection $N \cap HS_\star$ is normal in $HS_\star \cong M_{22}$, so by simplicity of M_{22} this intersection is either M_{22} or $\{1\}$. In the former case, $|N| = 100\,|N_\star| = |HS|$, which cannot happen because N is a strict subgroup. So $N \cap HS_\star = \{1\}$, and N has order 100.

By Sylow's theorems, if n_5 is the number of 5-Sylow subgroups of N then both $n_5 \mid (100/5^2) = 4$ and $n_5 \equiv 1 \bmod 5$ hold, so $n_5 = 1$. This Sylow subgroup $S \subset N$ is thus the unique subgroup of N of order 25, and because conjugation by elements in HS fixes N it also sends S to a 25-element subgroup of N, namely S. So S is normal in HS. But as shown above, any nontrivial normal subgroup of HS has order exactly 100, which is a contradiction. This proves that N does not exist, and HS is simple. □

4. Embedding in the Leech Lattice

The Leech lattice Λ_{24} is an even, unimodular, 24-dimensional lattice with minimal norm[1] 4, which may be constructed from the extended binary Golay code $\mathcal{G}$ as follows. For convenience, scale the standard Euclidean inner product by $\frac{1}{8}$ so that we may write the vectors of Λ_{24} with integer coordinates. The Leech lattice vectors have the form $(x_1, \ldots, x_{24})$ where each x_i is an integer such that

$$x_1 + \cdots + x_{24} \equiv 4x_1 \equiv \cdots \equiv 4x_{24} \bmod 8$$

and furthermore, for each residue $m \bmod 4$, the binary word with a 1 in the i^{th} slot exactly when $x_i \equiv m \bmod 4$ must be in the extended Golay code $\mathcal{G}$.

Recall that the automorphism group of the Golay code is M_{24}. Both M_{24} and the Golay code itself act on Λ_{24} by signed permutations: M_{24} naturally permutes the 24 coordinates, and a word $w \in \mathcal{G}$ negates the coordinates where w has a 1. This group $2^{12} \times M_{24}$ is not the full automorphism group $\mathrm{Aut}(\Lambda_{24})$, but it is useful for computing within the lattice. For example, we may use it to easily enumerate the minimal (norm 4) vectors of Λ_{24}. Suppose first that each x_i is even. If the Golay codeword of $2 \bmod 4$ elements is the zero codeword then all entries are divisible by 4 and their sum is divisible by 8. So there must be at least two nonzero coordinates, and the smallest possibility has shape $(\pm 4, \pm 4, 0^{22})$, with squared norm 4. Any permutation and sign choice is contained in Λ_{24}, and these comprise a single orbit under the action of $2^{12} \times M_{24}$. This orbit contains exactly $\binom{24}{2} \cdot 2^2 = 1104$ vectors of this shape. If the Golay codeword is nonzero, then there are at least 8 entries that are $2 \bmod 4$, and the smallest possibility has the form $(\pm 2^8, 0^{16})$ (and squared norm 4), where the eight ± 2s form a block of the 5-$(24, 8, 1)$ design and the number of minus signs is even. There are $759 \cdot 2^7 = 97152$ such vectors, again forming a single orbit under $2^{12} \times M_{24}$. Finally, if each of the coordinates is odd then not all of them can be ± 1, so the smallest possibility has 23 entries of ± 1 and one ± 3. It may be seen that all vectors in Λ_{24} of this form arise from the orbit of the norm 4 vector $(-3, 1^{23})$ under $2^{12} \times M_{24}$, and that this orbit has size $24 \cdot 2^{12} = 98304$. There are thus $1104 + 97152 + 98304 = 196560$ vectors in Λ_{24} of norm 4.

Let us show that the Higman-Sims graph may be found within Λ_{24}. Choose two vectors $v, w \in \Lambda_{24}$ of norm 3 such that $v - w$ has norm 2, for example

$$v = (5, 1, 1^{22}), \qquad w = (1, 5, 1^{22}).$$

It is known that the group $\mathrm{Aut}(\Lambda_{24})$ acts transitively on ordered pairs (v, w) with this property (see [5, §5.4]), so there is no loss of generality in choosing v, w as above. How many vectors $t \in \Lambda_{24}$ of norm 2 have

[1]In this section, norm always means squared length.

inner product 3 with both v and w? Equivalently, $t - v$ and $t - w$ must both have norm 2. Let T denote the set of such vectors. The copy of $M_{22} \subset M_{24}$ that stabilizes the first two coordinates acts on T, because such permutations do not affect distances to v and w.

Because we know exactly what the norm-2 vectors of Λ_{24} look like, we may easily enumerate T. If x is in the $2^{12} \times M_{24}$ orbit of $(\pm 4, \pm 4, 0^{22})$ then it must be the single vector $\star = (4, 4, 0^{22})$. If x has shape $(-3, 1^{23})$ (up to sign) then it must have a -3 in the last 22 coordinates with all other coordinates $+1$, so these vectors are in the M_{22} orbit of $(1, 1, -3, 1^{21})$ of size 22; call this orbit X. Finally, if it is in the $2^{12} \times M_{24}$ orbit of $(2^8, 0^{16})$ then it must be in the M_{22} orbit of $(2, 2, 2^6, 0^{16})$, of which there are 77 members; call this orbit $\mathcal{B}$. So T has exactly $1 + 22 + 77 = 100$ vectors.

Form the graph on T by creating edge (t_1, t_2) exactly when t_1 and t_2 have inner product 1, *i.e.* $t_1 - t_2$ has norm 2. This graph is isomorphic to the Higman-Sims graph. For proof, identify a vector in the X orbit with the coordinate containing -3, and identify a vector in the $(2, 2, 2^6, 0^{16})$ orbit $\mathcal{B}$ with the block of six coordinates (other than the first two) containing its 2s. Then $\star$ has inner product 1 with each $x \in X$ and inner product 2 with each $B \in \mathcal{B}$; any two $x_1, x_2 \in X$ have inner product 2; vectors $x \in X$ and $B \in \mathcal{B}$ have inner product 1 exactly when the -3 pairs with a 2, *i.e.* when B's six-tuple contains x's coordinate; and two vectors $B_1, B_2 \in \mathcal{B}$ have inner product 1 exactly when their six-tuples do not intersect. This completes the proof.

5. Connections with the Hoffman-Singleton Graph

Finally, we mention (without proof) some connections between the Higman-Sims graph and maximal cocliques in the Hoffman-Singleton graph G_{Hof}. The verification of some of these facts is illustrated in [2, §3.6] using the `GRAPE` computational package inside GAP, and the rest may be verified in the same way.

From its strongly regular parameters $(50, 7, 0, 1)$, the Hoffman-Singleton graph is seen to have eigenvalues $r = 2$ (multiplicity 28) and $s = -3$ (multiplicity 21) as well as 7 (multiplicity 1). It is known that in any k-regular graph on n vertices with minimal eigenvalue s, a maximal coclique has size at most $-ns/(k - s)$, which for G_{Hof} is $-(50)(-3)/(7 + 3) = 15$. It can be checked that G_{Hof} has exactly 100 cocliques of size 15, any two of which intersect in 0, 3, 5, 8, or 15 vertices. Form the graph on these 100 cocliques, where two cocliques are joined if they intersect in 0 or 8 vertices. The resulting graph is the Higman-Sims graph, as in [2]

The 100 cocliques in G_{Hof} are naturally divided into two equivalence classes of size 50, where two cocliques are in the same class if they intersect in 0, 5, or 15 vertices. Within each class, connecting disjoint cocliques creates a copy of the Hoffman-Singleton graph, and the above construction then shows that the Higman-Sims graph may be partitioned into two Hoffman-Singleton graphs; in fact, in [4] it is shown that there are exactly

352 such partitions. These equivalence classes also offer an alternative construction of G_{HS} from G_{Hof}: Form a graph on the 50 vertices of G_{Hof} and the 50 cocliques from one of the equivalence classes, where two vertices $v_1, v_2 \in V(G_{\mathrm{Hof}})$ are joined if they are connected in G_{Hof}, two cocliques are joined if they are disjoint, and a vertex $v \in V(G_{\mathrm{Hof}})$ connects to a coclique C precisely when $v \in C$. Then this graph is also the Higman-Sims graph.

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