Math Has This Funny Property

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Our Mathematical Minutiae article, “i Has This Funny Property” (this HCMR, p. 75), details the analytic power of the complex numbers \( \mathbb{C} \). Once we adjoin a single square root of \(-1\) to the real numbers \( \mathbb{R} \), we obtain startling results: Cauchy’s Theorem, Liouville’s Theorem, The Cauchy Integral Formula. . . .

But if we add just a few more square roots of \(-1\) to obtain the quaternions \( \mathbb{H} \supset \mathbb{C} \), some of the underlying structure breaks down. In the standard presentation, the roots \( \{i, j, k\} \) of \(-1\) do not even commute:

\[
ij = k = -ji, \quad jk = i = -kj, \quad ik = -j = -ki.
\]

Since \( \mathbb{H} \) is not commutative, it is not necessarily true that \( [q(z)]^n \) is “\( \mathbb{H} \)-analytic” when \( q(z) \) is. Thus, we figured, everything should break down once we pass from \( \mathbb{C} \) to \( \mathbb{H} \).

The endpaper would be called “\( j \) and \( k \) Have This Funny Property.” We would give all the standard counterexamples from quaternionic analysis, proving in a tour-de-force of mathematical irony how two new roots could devolve the entire analytic system of \( \mathbb{C} \) below its foundations.

We raced to the references. We read. We re-read. We—

We were wrong.

No counterexamples. In the late 1930s, Fueter obtained quaternionic analogues of Cauchy’s Theorem, Liouville’s Theorem, and even of power series developments. Our ironic tour-de-force was ruined before we were even born.

We despaired, until we thought to add four more square roots of \(-1\) to obtain the octonions \( \mathbb{O} \supset \mathbb{H} \supset \mathbb{C} \). We thought, this extension of \( \mathbb{C} \) is not even associative—there is no way octonionic analysis could be well-behaved!

We raced to the references. We read. We—

Math is funny like that.

†See biographical information in this HCMR, p. 75.