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ENDPAPER

Math Has This Funny Property

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Our Mathematical Minutiae article, "i Has This Funny Property" (this *HCMR*, p. 75), details the analytic power of the complex numbers $\mathbb{C}$. Once we adjoin a single square root of -1 to the real numbers $\mathbb{R}$, we obtain startling results: Cauchy's Theorem, Liouville's Theorem, The Cauchy Integral Formula. . . .

But if we add just a few more square roots of -1 to obtain the **quaternions** $\mathbb{H} \supset \mathbb{C}$, some of the underlying structure breaks down. In the standard presentation, the roots $\{i, j, k\}$ of -1 do not even commute:

$$ij = k = -ji, \quad jk = i = -kj, \quad ik = -j = -ki.$$

Since $\mathbb{H}$ is not commutative, it is not necessarily true that $[q(z)]^n$ is "$\mathbb{H}$-analytic" when $q(z)$ is. Thus, we figured, *everything should break down once we pass from* $\mathbb{C}$ *to* $\mathbb{H}$.

The endpaper would be called "j and k Have This Funny Property." We would give all the standard counterexamples from quaternionic analysis, proving in a tour-de-force of mathematical irony how two new roots could devolve the entire analytic system of $\mathbb{C}$ below its foundations.

We raced to the references. We read. We re-read. We—

We were wrong.

No counterexamples. In the late 1930s, Fueter obtained quaternionic analogues of Cauchy's Theorem, Liouville's Theorem, and even of power series developments. Our ironic tour-de-force was ruined before we were even born.

We despaired, until we thought to add four more square roots of -1 to obtain the **octonions** $\mathbb{O} \supset \mathbb{H} \supset \mathbb{C}$. We thought, *this extension of* $\mathbb{C}$ *is not even* associative—*there is no way octonionic analysis could be well-behaved!*

We raced to the references. We read. We—

Math is funny like that.

† See biographical information in this *HCMR*, p. 75.