Mean Geometry

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1 Introduction (for lack of a better section header)

There are generally two opposite approaches to Olympiad geometry. Some prefer to draw the diagram and simply stare (labeling points only clutters the diagram!), waiting for the interactions between the problem's various elements to present themselves visually. Others toss the diagram onto the complex or coordinate plane and attempt to establish the necessary connections through algebraic calculation rather than geometric insight.

This article discusses an interesting way to visualize and approach a variety of geometry problems by combining these two common methods: synthetic and analytic. We'll focus on a theorem known as "The Fundamental Theorem of Directly Similar Figures" [4] or "The Mean Geometry Theorem" (abbreviated here as MGT). Although the result is quite simple, it nevertheless encourages a powerful new point of view.

1.1 Basic Definitions

A figure has **positive orientation** if its vertices are listed in counterclockwise order (like pentagons $ABCDE$ or $LMNOP$ in diagram 1), and otherwise it has **negative orientation** (like $VWXYZ$ or even $EDCBA$). Two figures are **similar** if. . . , well, we're all familiar with similar figures; two similar figures are **directly similar** if they have the same orientation, and otherwise they're **inversely similar**. For example, pentagon $ABCDE$ is directly similar to pentagon $LMNOP$, but inversely similar to $VWXYZ$.

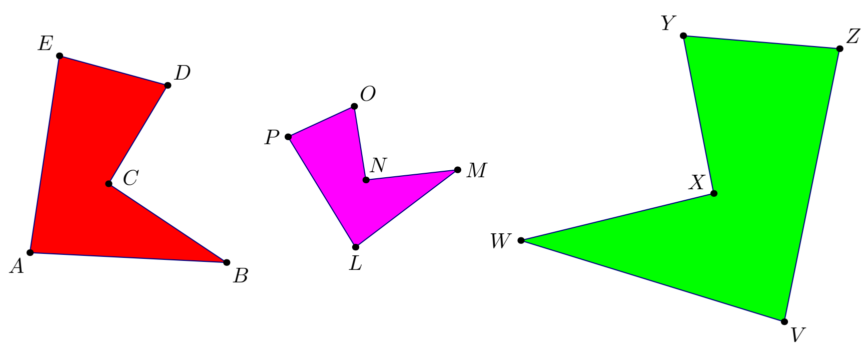

Figure 1: Direct and inverse similarity

1.2 Point Averages

Now for some non-standard notation. We define the **average of two points** A and B as the midpoint M of segment AB, and we write $\frac{1}{2}A + \frac{1}{2}B = M$. We can also compute **weighted averages** with weights other than $\frac{1}{2}$ and $\frac{1}{2}$, as long as the weights add to 1: the weighted average $(1-k)A + (k)B$ is the point X on line AB so that $AX/AB = k$. (This is consistent with the complex-number model of the plane, though we will still treat A and B as *points* rather than *complex values*.) For example, $N = \frac{2}{3}A + \frac{1}{3}B$ is one-third of the way across from A to B (see Figure 2).

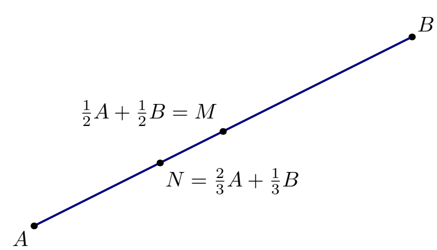

Figure 2: Point averaging

1.3 Figure Averages

We can use the concept of averaging points to define the **(weighted) average of two figures**, where a **figure** may be a polygon, circle, or any other 2-dimensional path or region (this article will stick to polygons). To average two figures, simply take the appropriate average of corresponding pairs of points. For polygons, we need only average corresponding vertices:

$$\frac{1}{2}A_i + \frac{1}{2}B_i = C_i \text{ for } i = 1,2,3,4, \text{ so}$$
$$\frac{1}{2}A_1A_2A_3A_4 + \frac{1}{2}B_1B_2B_3B_4 = C_1C_2C_3C_4.$$

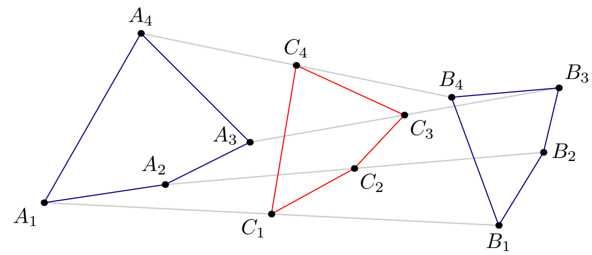

Figure 3: Figure averaging

1.4 Finally!

By now you can probably guess the punchline.

Mean Geometry Theorem. *The (weighted) average of two directly similar figures is directly similar to the two original figures.*

Look again at Figure 3. Quadrilaterals $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are directly similar, so the Mean Geometry Theorem guarantees that their average, $C_1C_2C_3C_4$, is directly similar to both of them.

2 Proofs of MGT

We mentioned in the introduction that this article incorporates some ideas from both synthetic and analytic geometry. Here we offer two proofs of the Theorem – one based in each category – to better illustrate the connection.

Although MGT holds for all types of figures, it suffices to prove result for triangles (why?):

MGT for Triangles. *If $\triangle_a = A_1A_2A_3$ and $\triangle_b = B_1B_2B_3$ are directly similar triangles in the plane, then the weighted average $\triangle_c = (1-k)A_1A_2A_3 + (k)B_1B_2B_3 = C_1C_2C_3$ is similar to both $\triangle_a$ and $\triangle_b$ (Figure 4).*

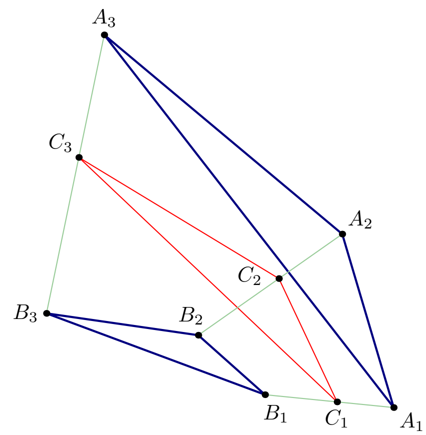

Figure 4: MGT for triangles

2.1 Proof 1: Spiral Similarity

If $\triangle_b$ is simply a translation of $\triangle_a$, say by vector $\overrightarrow{\mathbf{V}}$, then the translation by vector $k\overrightarrow{\mathbf{V}}$ sends $\triangle_a$ to $\triangle_c$, so they are directly similar (in fact congruent).

Otherwise, there exists a unique spiral similarity[1] ${}^{r}_{\theta}O$ that takes $\triangle_a$ to $\triangle_b$ (see [5, Theorem 4.82]). The three triangles A_iOB_i (for $i = 1, 2, 3$) have $\angle A_iOB_i = \theta$ and $OB_i/OA_i = r$, so they are all similar (see Figure 5).

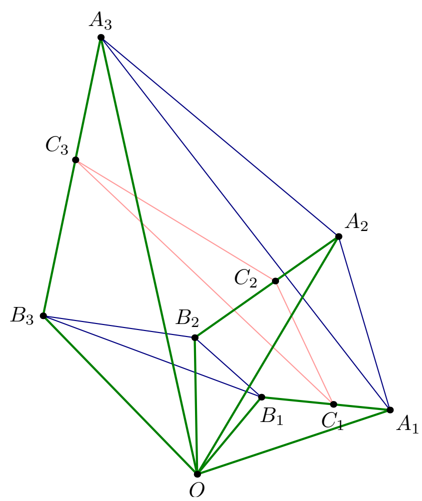

Figure 5: Similar triangles A_iOB_i

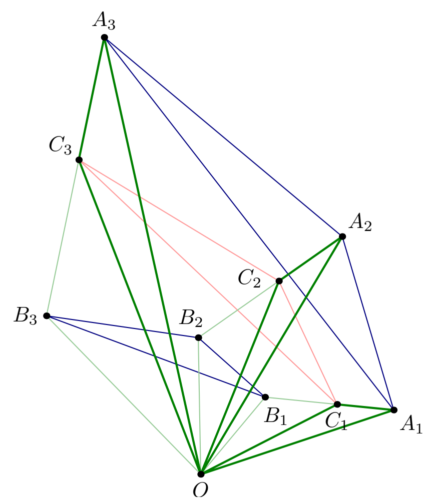

Figure 6: Similar triangles A_iOC_i

Since C_1, C_2, and C_3 are in corresponding positions in these three similar triangles (since $A_iC_i/A_iB_i = k$ for $i = 1, 2, 3$), the three triangles A_iOC_i ($i = 1, 2, 3$) are similar to each other (Figure 6). Thus the spiral similarity ${}^{r'}_{\theta'}O$, where $\theta' = \angle A_1OC_1$ and $r' = OC_1/OA_1$, sends $\triangle A_1A_2A_3$ to $\triangle C_1C_2C_3$, so these triangles are directly similar, as desired.

2.2 Proof 2: Complex Numbers

We'll use capital letters for points and lower case for the corresponding complex numbers: point A_1 corresponds to the complex number a_1, and so on.

[1] i.e. a rotation through an angle θ followed by a dilation with ratio r, both centered at the point O

Since the triangles are similar, $\angle A_2A_1A_3 = \angle B_2B_1B_3$ and $A_1A_3/A_1A_2 = B_1B_3/B_1B_2$. If z is the complex number with argument $\angle A_2A_1A_3$ and magnitude A_1A_3/A_1A_2, we have $\frac{a_3-a_1}{a_2-a_1} = z = \frac{b_3-b_1}{b_2-b_1}$. Or, rearranged,

$$a_3 = (1-z)a_1 + (z)a_2 \quad \text{and} \quad b_3 = (1-z)b_1 + (z)b_2. \tag{$\clubsuit$}$$

To prove the theorem, we need to show that $\triangle C_1C_2C_3$ behaves similarly (no pun intended), i.e. we need to show that $c_3 = (1-z)c_1 + (z)c_2$. But since $c_i = (1-k)a_i + (k)b_i$ for $i = 1, 2, 3$ (by definition of $\triangle_c$), this equality follows directly from ($\clubsuit$):

$$\begin{aligned} c_3 &= (1-k)a_3 + (k)b_3 \\ &= (1-k)\big((1-z)a_1 + (z)a_2\big) + (k)\big((1-z)b_1 + (z)b_2\big) \\ &= (1-z)\big((1-k)a_1 + (k)b_1\big) + (z)\big((1-k)a_2 + (k)b_2\big) \\ &= (1-z)c_1 + (z)c_2, \end{aligned}$$

as desired.

3 Some Problems

The theorem has been proved, but there might be some lingering doubts about the usefulness of such a seemingly simple and specialized statement.[2] In this section, we'll put the Theorem to work, and we'll learn to recognize when and how MGT may be applied to a variety of problems.

3.1 Equilaterals Joined at the Hip

Problem 1 (Engel)**.** *OAB and OA_1B_1 are positively oriented regular triangles with a common vertex O. Show that the midpoints of OB, OA_1, and AB_1 are vertices of a regular triangle. (Remember:* **positively oriented** *means the vertices are listed in counterclockwise order.)*

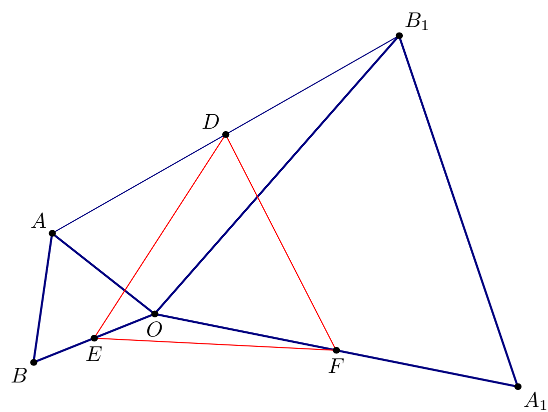

Figure 7: Problem 1

To show that DEF is equilateral, we'd like to express $\triangle DEF$ as an average of two other *directly similar* equilateral triangles. From the diagram we see that

$$\frac{1}{2}A + \frac{1}{2}B_1 = D, \qquad \frac{1}{2}B + \frac{1}{2}O = E, \quad \text{and} \quad \frac{1}{2}O + \frac{1}{2}A_1 = F,$$

and putting these together yields

$$\frac{1}{2}ABO + \frac{1}{2}B_1OA_1 = DEF.$$

So we're done by MGT, right?

Let's make sure everything is in place. Are triangles ABO and B_1OA_1 directly similar? The problem tells us that they are both positively oriented equilateral triangles, so yes. Are we indeed taking a weighted average of the two triangles? In other words, do the weights add to 1? Of course! $\frac{1}{2} + \frac{1}{2} = 1$.

Now we can confidently apply the Mean Geometry Theorem and conclude that triangle DEF must be directly similar to $\triangle ABO$ and $\triangle B_1OA_1$, i.e. $\triangle DEF$ is equilateral. ∎

3.2 Napoleon's Last Hurrah

Problem 2 (Napoleon's Theorem)**.** *If equilateral triangles BCP, CAQ, ABR are erected externally on the sides of triangle ABC, their centers X, Y, Z form an equilateral triangle (Figure 8).*

[2] Please pardon the unintentional alliteration.

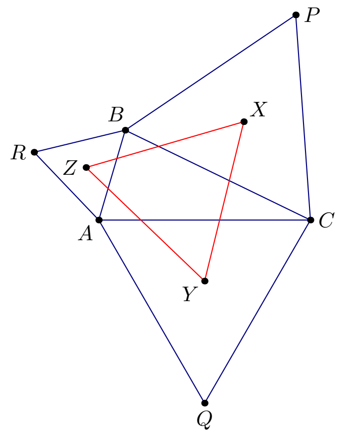

Figure 8: Napoleon's Theorem

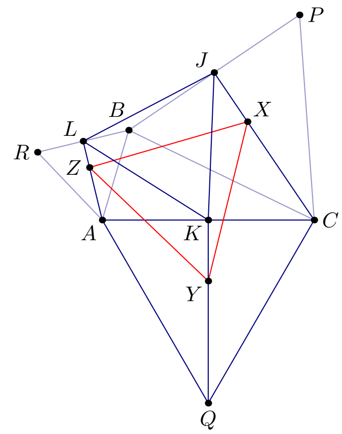

Figure 9: Equilateral triangle JKL

The setup looks a lot like the configuration in the first problem, where two equilaterals share a common vertex. We can try to mimic this earlier configuration by considering equilateral triangle $\frac{1}{2}PCB + \frac{1}{2}BAR = JKL$. (Figure 9)

Now the way to reach the target points, X, Y, and Z, presents itself: X is on median JC, and it's $\frac{1}{3}$ of the way across. This means that $X = \frac{2}{3}J + \frac{1}{3}C$, and likewise for Y and Z. We can write

$$XYZ = \frac{2}{3}JKL + \frac{1}{3}CQA,$$

and since both $\triangle JKL$ and $\triangle CQA$ are negatively oriented equilateral triangles, we're done!

3.3 Extending MGT?

Let's look closer at that solution. First we averaged PCB and BAR: $\frac{1}{2}PCB + \frac{1}{2}BAR = JKL$. Then, we averaged this with CQA: $\frac{2}{3}JKL + \frac{1}{3}CQA = XYZ$. Momentarily indulging ourselves in some questionable manipulation, we can substitute the first equation into the second and simplify to find

$$\frac{1}{3}PCB + \frac{1}{3}BAR + \frac{1}{3}CQA = XYZ. \qquad (\spadesuit)$$

This suggests that we may be able to generalize MGT to three figures, like so:

Extended MGT. *Define the* **weighted average of 3 points** $\omega_1 P_1 + \omega_2 P_2 + \omega_3 P_3$ *(where the weights ω_1, ω_2, and ω_3 are real numbers and add to 1) just as we would in the complex plane.*[3] *Then if $\triangle A_1A_2A_3$, $\triangle B_1B_2B_3$, and $\triangle C_1C_2C_3$ are directly similar triangles, their weighted average*

$$\omega_a A_1A_2A_3 + \omega_b B_1B_2B_3 + \omega_c C_1C_2C_3$$

is also directly similar to them. This naturally extends to include figures other than triangles (as long as they're all similar to each other) or more than 3 similar figures (as long as the weights add to 1).

Can you prove this? Anyway, by this generalized version of MGT, equation ($\spadesuit$) alone provides a succinct, one-line proof of Napoleon's Theorem. (Cool, huh?)

3.4 But Wait, There's More!

Problem 3. *Napoleon isn't done with us yet: prove that triangles ABC and XYZ have the same centroid.*

[3] As a notable special case, $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$ is the centroid of triangle $P_1P_2P_3$.

Like Napoleon's Theorem itself, this can also be done in one line. Here's the first half:

$$\frac{1}{3}BXC + \frac{1}{3}CYA + \frac{1}{3}AZB = ??? \qquad (\heartsuit)$$

Before you read ahead, try to figure out how and why this proves the problem. (Jeopardy song plays...) Ok, welcome back!

The three red triangles are isosceles $30-120-30$ triangles, so they are all similar. If G and H denote the centroids of $\triangle ABC$ and $\triangle XYZ$ respectively, then the result of the expression in equation ($\heartsuit$) is triangle GHG, which (by the Theorem) must also be a $30-120-30$ triangle. But two of its vertices are at the same place! The triangle has thus degenerated into a point, so *all three* of its vertices are at the same place, and $G = H$.

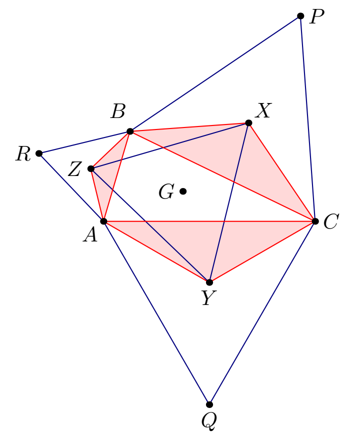

Figure 10: More Napoleon

3.5 I Can't Take Any More Equilaterals! and the Asymmetric Propeller

Let's do one more problem of a similar flavor before we move on.

Problem 4. *Positively oriented equilateral triangles XAB, XCD, and XEF share a vertex X. If P, Q, and R are the midpoints of BC, DE, and FA respectively, prove that PQR is equilateral.*

Problem 5 (Crux Mathematicorum)**.** *In quadrilateral $ABCD$, M is the midpoint of AB, and three equilateral triangles BCE, CDF, and DAG are constructed externally. If N is the midpoint of EF and P is the midpoint of FG, prove that MNP is equilateral.*

Problem 6. *The four triangles ABC, AA_bA_c, B_aBB_c, and C_aC_bC are directly similar, and M_a, M_b, and M_c are the midpoints of B_aC_a, C_bA_b, and A_cB_c. Show that $M_aM_bM_c$ is also similar to ABC.*

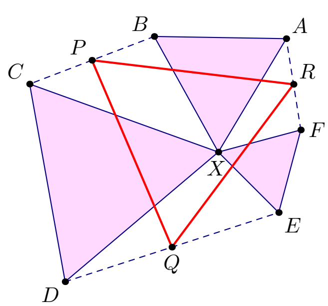

Figure 11: Problem 4

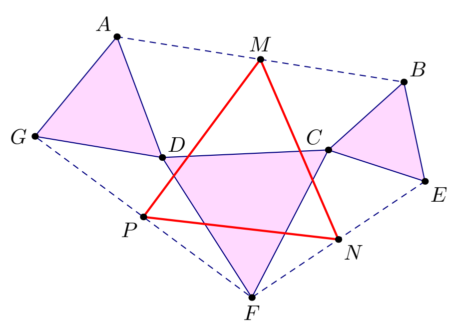

Figure 12: Problem 5

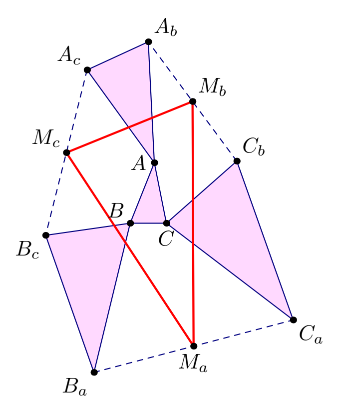

Figure 13: Problem 6

Hm, didn't I just say *one* more problem?

Indeed, all we have to do is solve problem 6; the rest come free. The diagrams are drawn to illustrate illustrate how the first two problems are special cases of problem 6.[4] So, let's solve problem 6, known as the Asymmetric Propeller [2].

The problem gives us a plethora of similar triangles to work with, so our first instinct should be to try to write triangle $M_aM_bM_c$ as an average of these:

$$M_aM_bM_c = \omega ABC + \omega_a AA_bA_c + \omega_b B_aBB_c + \omega_c C_aC_bC. \qquad (\diamondsuit)$$

[4] In problem 4, the middle triangle has degenerated into point X. In problem 5, one of the outside triangles degenerates into point F.

The three triangles AA_bA_c, B_aBB_c, and C_aC_bC play identical roles in the problem, so we have every reason to guess that $\omega_a = \omega_b = \omega_c$. Now, for ($\diamondsuit$) to work, we need

$$M_A = \frac{1}{2}(B_a + C_a) = (\omega + \omega_a)A + \omega_a(B_a + C_a).$$

If we set $\omega_a = \frac{1}{2}$ and $\omega = -\omega_a = -\frac{1}{2}$, then this works out perfectly (and likewise for M_b and M_c). So does

$$M_aM_bM_c = -\frac{1}{2}ABC + \frac{1}{2}AA_bA_c + \frac{1}{2}B_aBB_c + \frac{1}{2}C_aC_bC$$

finish the proof?[5] The triangles are all similar and the weights add to 1 ($= -\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$), so yes. Sweet, another one liner. ∎

[5]Remember that negative weights are allowed, as long as the weights add to 1.

4 Figure Addition

I keep stressing that the weights in a weighted average must add to 1. What would happen if they didn't?

In the complex-number proof of MGT from section 2.2, there's really no reason the weights must be k and $(1-k)$. They could be *any* real numbers, and the proof works the same! There is one subtle difference, though. In the diagram to the right, use weights of $\omega_1 = \omega_2 = 1$, and we'll try to calculate $A+B$. Where is it?

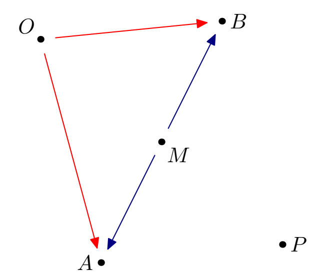

Figure 14: Point addition

To add vectors or complex numbers like this, we need to know where the origin is. If the origin is at, say, M, the midpoint between A and B, then $A+B$ represents the sum of the blue vectors $\overrightarrow{\mathbf{MA}} + \overrightarrow{\mathbf{MB}} = \overrightarrow{\mathbf{0}} = M$. But if we put the origin at a different point O, then the sum $A+B$ is now the sum of the red vectors $\overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{OB}} = \overrightarrow{\mathbf{OP}} = P$. So, with **point addition**, the sum depends on the location of the origin, i.e. *we must first specify an origin.* With this minor change, MGT extends yet again:

MGT: Figure Addition. *For real numbers ω_1 and ω_2, define the* **sum of points** $\omega_1 P_1 + \omega_2 P_2$ *as the endpoint of the vector* $\omega_1 \overrightarrow{\mathbf{OP_1}} + \omega_2 \overrightarrow{\mathbf{OP_2}}$*, where O is a specified origin. Then if $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$ are directly similar, the* **sum of figures** $\omega_a A_1A_2A_3 + \omega_b B_1B_2B_3$ *(formed by adding corresponding vertices) is directly similar to the original two triangles. As before, this naturally extends to more complicated figures and to more than two figures.*

There is another way to visualize figure addition through dilation. For two similar triangles $A_1B_1C_1$ and $A_2B_2C_2$, the sum $A_1B_1C_1 + A_2B_2C_2$ may be constructed by averaging the two triangles, $\frac{1}{2}A_1B_1C_1 + \frac{1}{2}A_2B_2C_2$, and then dilating this with center O and ratio 2.

4.1 When Did the IMO Get So Easy? (answer: 1977)

Problem 7 (IMO jury 1977 [6])**.** *OAB and $OA'B'$ are regular triangles of the same orientation, S is the centroid of $\triangle OAB$, and M and N are the midpoints of $A'B$ and AB', respectively. Show that $\triangle SMB' \sim \triangle SNA'$ (Figure 15).*

(In the diagram, points W, X, Y, and Z are midpoints.)

First of all, the red triangles both look like $30-60-90$ triangles, and if we can prove that, we're done. The two equilateral triangles give us plenty of $30-60-90$s to work with; all we have to do is find the right ones!

Points O, Z, N, and W form a parallelogram, so if O is the origin, $Z+W = N$. With a little experimentation, we arrive at $NSA' = ZOA' + WSO$, and since both $\triangle ZOA'$ and $\triangle WSO$ are positively oriented $30-60-90$ triangles, so is $\triangle NSA'$. Similarly, $MSB' = YOB' + XSO$ (again with O as origin) proves that $\triangle MSB'$ is a negatively oriented $30-60-90$ triangle. Whew, that was quick! ∎

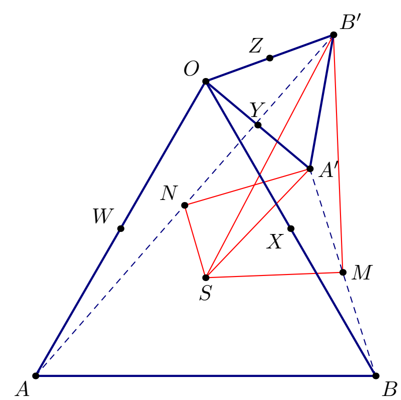

Figure 15: Problem 7

4.2 A Fresh Look at an Old Result (1936, to be precise)

Problem 8 (Pompeiu's Theorem [1])**.** *Given an equilateral triangle ABC and a point P that does not lie on the circumcircle of ABC, one can construct a triangle of side lengths equal to PA, PB, and PC. If P lies on the circumcircle, then one of these three lengths is equal to the sum of the other two.*

Erect equilateral triangles PCY and BPX with the same orientation as $\triangle ABC$. With P as origin, consider equilateral triangle $PCY + BPX = BCA'$. It must be equilateral with the same orientation as ABC, which

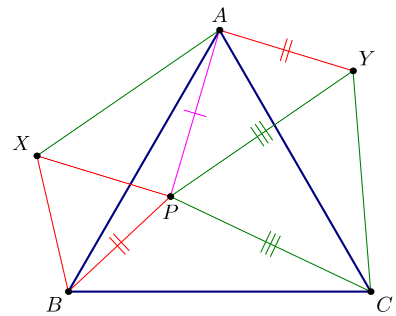

Figure 16: Pompeiu's Theorem

means $A' = Y + X = A$, i.e. $PYAX$ is a parallelogram. Notice that $\triangle APY$ has $AP = AP$, $PY = CP$, and $YA = PX = BP$, so if it is not degenerate, $\triangle APY$ is the triangle we're looking for.

When is this triangle degenerate, i.e. when are A, P and Y collinear? This happens if and only if (using directed angles modulo $180°$) $\angle CPA = \angle CPY = 60° = \angle CBA$, i.e. quadrilateral $ABCP$ is cyclic. And certainly, if $\triangle APY$ is degenerate, then one of its sides equals the sum of the other two. ◪

5 A Few Harder Problems

At this point we've become proficient in utilizing the various forms of the Theorem, and we've learned to look for a MGT approach when the problem hands us loads of similar triangles with which to play. But MGT can be useful in many other situations as well, even if its application may be far from obvious. In this section, we'll look at a few harder problems to stress that *the MGT may not solve every problem immediately, but it's a valuable method to keep in mind as you explore a problem.* It may be that MGT is only one of many steps in your solution. Or, ideas related to MGT may lead you to a solution that doesn't use it at all. The point is that looking at a diagram from an MGT viewpoint may tell you things that you formerly wouldn't have noticed.

This brings up another point: in order to recognize uses for MGT, you *must* have an accurate diagram – or two, or three – to look at. This article is filled with diagrams for exactly that purpose. (Whether you're solving a geometry problem using MGT or not, it's usually a good idea to have a decent diagram handy!)

On that note, let's bring on the problems.

5.1 A Pretty(,) Busy Diagram

Problem 9 (IMO Shortlist 2000, G6)**.** *Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, and let X be a point inside $ABCD$ such that $\angle ADX = \angle BCX < 90^\circ$ and $\angle DAX = \angle CBX < 90^\circ$. If Y is the point of intersection of the perpendicular bisectors of AB and CD, prove that $\angle AYB = 2\angle ADX$.*

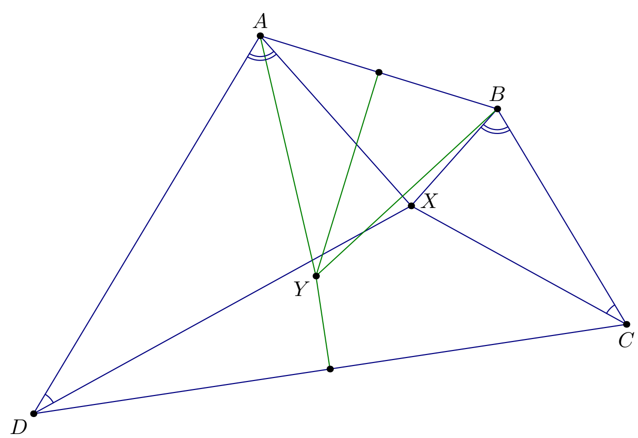

Figure 17: Problem 9

First thing to notice: by simply relabeling the diagram, it must also be true that $\angle DYC = 2\angle DAX$. So, designate $\angle ADX = \alpha$ and $\angle DAX = \delta$. Next, notice that triangles AXD and BXC are similar. *Ooh, that means we should form another similar triangle $\frac{1}{2}AXD + \frac{1}{2}BXC$!* That's a decent thought, but unfortunately, even though AXD and BXC are similar, they're not *directly* similar.

It may not clear where to go from here, but since it's necessary to start *somewhere*, we'll begin with the only tangible fact we have: similar triangles AXD and BXC. It's interesting that no matter how these two triangles are hinged or scaled around X, point Y still has its curious property. This suggests a possible direction for exploration: scale the triangles and see what happens to point Y.

Let's leave triangle AXD fixed while we enlarge and shrink BXC. What size triangle would be easier to analyze? How about zero! Consider quadrilateral $AB'C'D$, where $\triangle B'XC'$ has shrunken to the degenerate triangle at point X (Figure 18). The corresponding Y' is the intersection of the perpendicular bisectors of $AB' = AX$ and $DC' = DX$, i.e. the circumcenter O of triangle AXD. Is it true that $2\angle ADX = \angle AOX$? Yes, since arc $\widehat{AX}$ of circle O has measure 2α. So when triangle BXC shrinks to zero, everything works out as expected.

What other size for triangle BXC might work well? Let's look at quadrilateral $AB''C''D$, where $B''XC''$ is *congruent* to AXD as shown in Figure 19. To locate Y'', we should be looking for the perpendicular bisectors

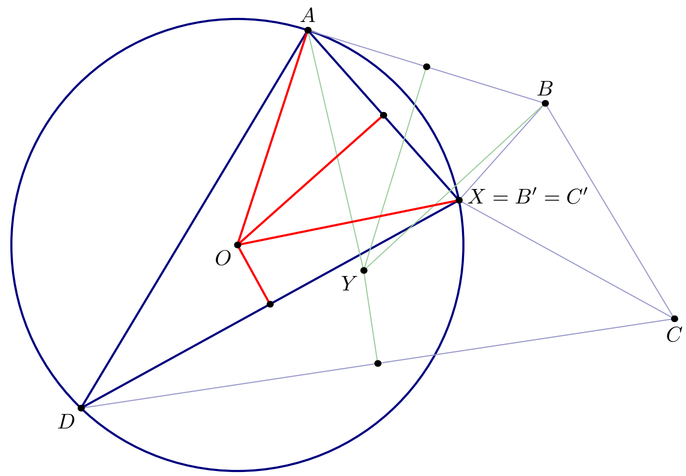

Figure 18: Studying "quadrilateral" $AB'C'D$

of AB'' and $C''D$. But these are the same line $\overleftrightarrow{MN}$, so where is Y''? We can instead locate Y'' by finding the point on this line so that $\angle AY''B'' = 2\alpha$, since this is another property our Ys should have. This means that $\angle AY''X = \alpha = \angle ADX$, so $AXY''D$ is cyclic, i.e. Y'' is the second intersection of circle O with line MN. (If the circle happens to be tangent to line MN, 'second intersection' simply means tangency point.)

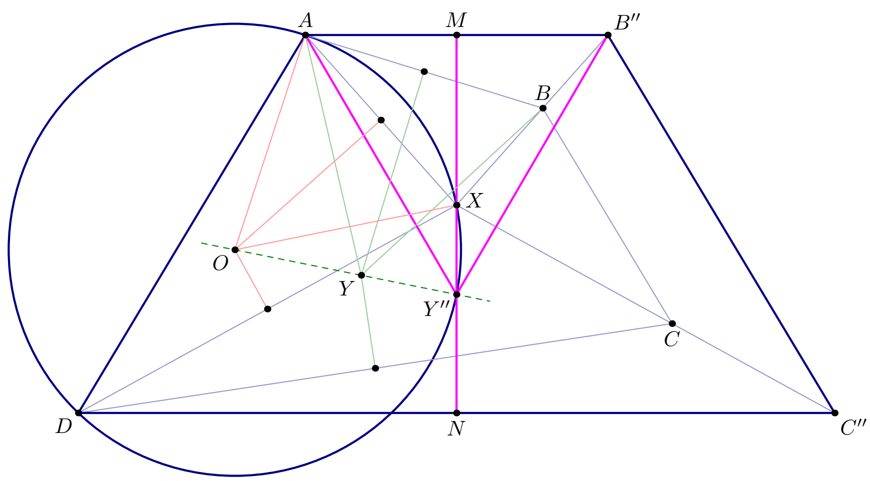

Figure 19: Quadrilateral $AB''C''D$

It seems that we've lost sight of our original problem. We've studied quadrilaterals $AXXD$ and $AB''C''D$, but not the original $ABCD$. Luckily, a perusal of diagram 19 reveals the next step: triangles AOX, $AY''B''$, and AYB are similar since they're all isosceles with vertex angle 2α, so we should be able to average them. Indeed, if $XB/XB'' = k$, then we should have $(1-k)AOX + (k)AY''B'' = AYB$. This is (almost) the last step to a complete solution! A full, self-contained solution is given below.

Full Solution. Let $XB/XA = k$, and dilate B and C around X with ratio $1/k$ to points B'' and C'' respectively, so that $AB''C''D$ is an isosceles trapezoid. M and N are the midpoints of AB'' and $C''D$. Define O as the circumcenter of triangle AXD, and let Y'' be the second intersection of circle ADX with line MN. If $\angle ADX = \alpha$ and $\angle DAX = \delta$, it follows that $\angle AOX = 2\alpha$, $\angle AY''X = \alpha$ which implies $\angle AY''B'' = 2\alpha$, and likewise, $\angle DOX = 2\delta$ and $\angle DY''C'' = 2\delta$. Therefore, the isosceles triangles AOX and $AY''B''$ are similar, as are triangles DOX and $DY''C''$.

Define $Y_1 = (1-k)O + (k)Y''$, and notice that triangles $AY_1B = (1-k)AOX + (k)AY''B''$ and $DY_1C =$

$(1-k)DOX + (k)DY''C''$ are isosceles by MGT. Thus, Y_1 is the intersection of the perpendicular bisectors of AB and CD, i.e. $Y_1 = Y$. Furthermore, by our application of MGT, $\angle AYB = \angle AOX = 2\alpha$, QED.

5.2 Hidden Circles

Problem 10 (USA TST 2005 #6)**.** *Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.*

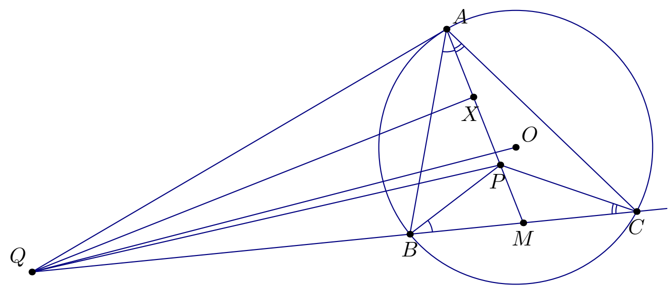

Figure 20: Problem 10

Most of the diagram is straightforward: O is the circumcenter, Q is the intersection of BC with the perpendicular bisector of AP, and I've added M, the intersection of AP with BC. Everything is simple to navigate except P itself, so that's where we'll start investigating.

The first strange angle equality, i.e. $\angle PAB = \angle PBC$, shows that $\triangle MAB \sim \triangle MBP$. Thus, $MA/MB = MB/MP$, or $(MA)(MP) = (MB)^2$. This shows, by power of a point, that the circle through A, P, and B is tangent to line MB at B.[6] Likewise, the circle through A, P, and C is tangent to MC at C. (Call these two circles [and their centers] O_1 and O_2 respectively.) Finally, $(MB)^2 = (MA)(MP) = (MC)^2$, so M is the midpoint of BC.

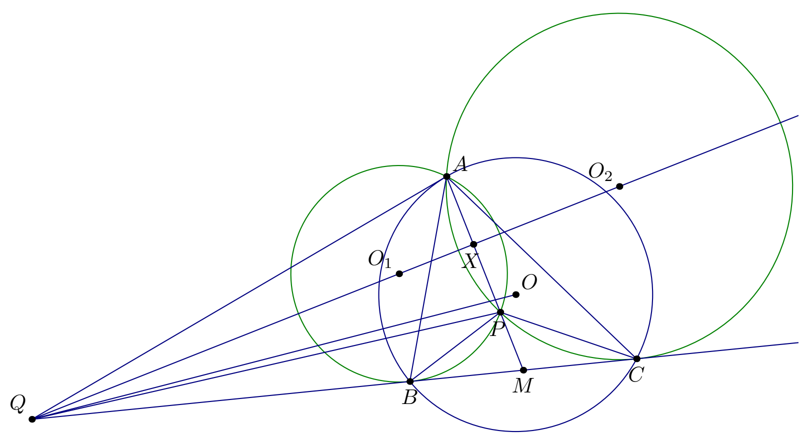

Figure 21: Circles O_1 and O_2

The diagram becomes much clearer from the viewpoint of the two circles. The circles intersect at A and P, and line BC is their common tangent. Point Q, being the intersection of their axis of symmetry (line O_1O_2) with a common tangent (line BC), must be the center of homothecy between circles O_1 and O_2.

Let X be the midpoint of AP. We're asked to prove that $\angle AQP = 2\angle OQB$, or equivalently, $\angle AQX = \angle OQM$. But triangles AQX and OQM are both right triangles, so we need to prove that they are similar. Here's where

[6] If this circle intersected line MB at some other point B', then power of a point would show that $(MB)^2 = (MA)(MP) = (MB)(MB')$, so $MB = MB'$ and $B = B'$.

Mean Geometry might come in handy: to prove that these triangles are similar, we might be able to find a third triangle similar to, say $\triangle AQX$, and then express triangle OQM as a weighted average of those two. So now we hunt for a third similar triangle. In order for this to work, this triangle should have one vertex at Q, one vertex along line XM, and one vertex along line AO. What key points are on these two lines? Just A and P. And since A is already being used, let's think about P. If triangles AQX and OQM were similar to triangle ZQP, where would this mystery point Z have to be? We already want it on line AO, and since $\angle AQX = \angle XQP = \angle ZQP$, we would also need Z to be on line QX. So define $Z = AO \cap QX$, and let's see if $\triangle ZQP$ is indeed the triangle we're looking for. First of all, is it a right triangle?

Angle QPZ is a right angle if and only if $\angle QAZ$ is right, so we need $QA \perp OA$, i.e. QA should be tangent to circle O at A. It turns out to be true, as follows. Let T be the second intersection of QA with circle O_1. If ${}^{r}Q$ (where $r = QO_2/QO_1$) is the homothecy taking circle O_1 to circle O_2, we have ${}^{r}Q(T) = A$ and ${}^{r}Q(B) = C$, so $\angle ACB = \angle TBQ = \angle TAB$. Thus, triangles QBA and QAC are similar, so $(QB)(QC) = (QA)^2$, and QA is tangent to circle O. So (by tracing backwards through a few lines of reasoning above), triangle ZQP actually *does* have a right angle at P. This means it is similar to triangle AQX.

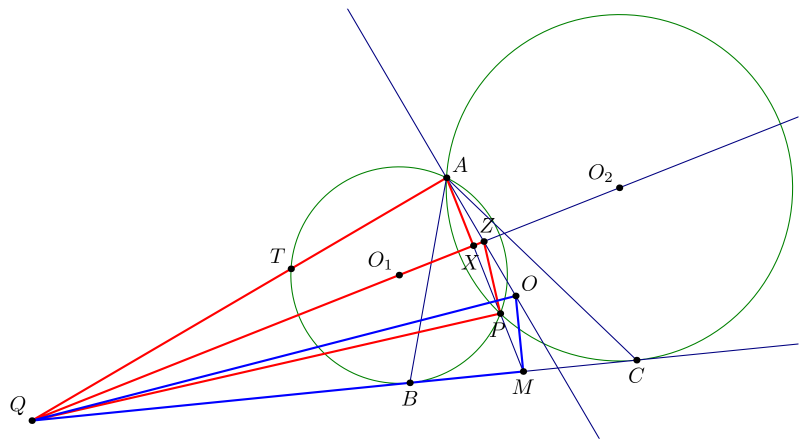

Figure 22: Three similar triangles

The next part of our initial plan was to find a weighted average of triangles ZQP and AQX that would produce triangle OQM, thus completing the proof. So, all we need to show is that $AO/ZO = XM/PM$. Good luck.

Those segment lengths aren't easy to calculate, even with plenty of paper and tons of time. For the first time in this article, MGT fails to miraculously save the day. But we've come far enough with the MGT idea so that the proof is moments away. Let ${}^{r}_{\theta}Q$ be the spiral dilation centered at Q that sends XP to AZ. Since M is on line XP, ${}^{r}_{\theta}Q(M) = M'$ is on line ZA. Also, since triangle $M'QM$ is similar to triangle AQX, we have $\angle QMM' = 90°$, so M' is also on the perpendicular bisector of BC. This means ${}^{r}_{\theta}Q(M) = M' = O$. Thus, $\angle OQM = \angle AQX$, as desired. (Notice that, even though MGT was a driving force for most of the solution, not a single mention of it is necessary in the final writeup.)

5.3 Nagel Who?

Problem 11 (USAMO 2001 #2)**.** *Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.*

In the diagram I've added F_1 and F_2 to preserve symmetry, and I included the midpoints of the sides of the triangle.

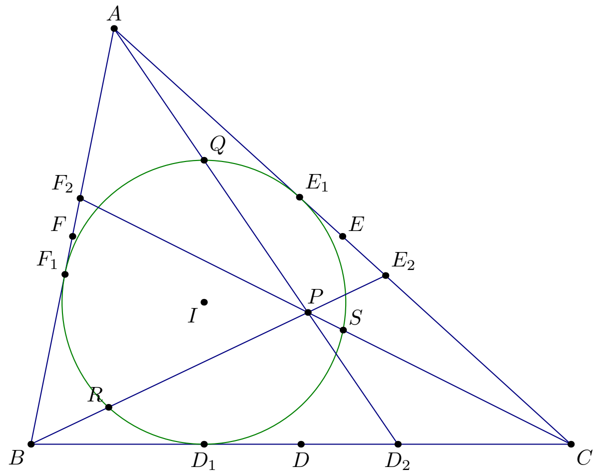

Figure 23: Problem 11

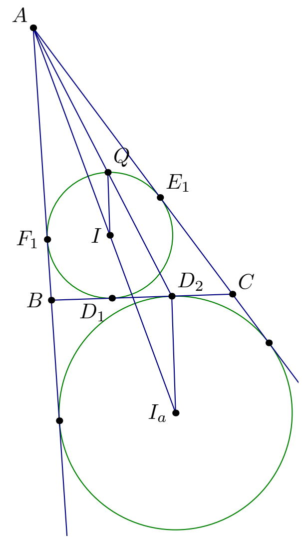

Figure 24: Excircle I_a

Using the standard notations $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$, it is relatively well known that $BD_1 = s - b$. Indeed, if $AE_1 = AF_1 = x$, $BF_1 = BD_1 = y$, $CD_1 = CE_1 = z$, then the system $y + z = a$, $z + x = b$, $x + y = c$ can be solved to give $BD_1 = y = (c + a - b)/2 = s - b$. Likewise, if the excircle opposite A is tangent to BC at V, a similar calculation shows that $CV = s - b = BD_1$, i.e. $V = D_2$. So D_2, E_2, F_2 are the tangency points of BC, CA, AB with the triangle's three excircles.

Another relatively well-known point in the diagram is P, commonly referred to as the **Nagel point** of triangle ABC. It's simply the intersection of the three cevians AD_2, BE_2, and CF_2, which must concur by Ceva's theorem:

$$\frac{BD_2}{D_2C} \cdot \frac{CE_2}{E_2A} \cdot \frac{AF_2}{F_2B} = \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} \cdot \frac{s-b}{s-a} = 1.$$

A useful property of the **Nagel point**,[7] other than its mere existence, is how its defining cevians interact with the incircle: line AD_2 intersects the incircle at point Q *diametrically opposite from* D_1. To prove this, let $r = AI/AI_a$ (in Figure 24), so that ${}^rA$ is the homothecy centered at A taking excircle I_A to incircle I. Since D_2I_A is perpendicular to BC, its image through ${}^rA$, namely QI, is also perpendicular to BC. Therefore, QD_1 is a diameter of circle I. The same goes for diameters E_1R and F_1S.

Now to the problem. The midpoints D, E, and F inspire us to consider

$$\frac{1}{2}\triangle D_1E_1F_1 + \frac{1}{2}\triangle D_2E_2F_2 = \triangle DEF. \qquad (\bigstar)$$

Triangles $D_1E_1F_1$ and $D_2E_2F_2$ aren't similar, so MGT doesn't tell us anything directly. But it's still worth noticing.

If it's true that $AQ = D_2P$, it must also happen that $BR = E_2P$ and $CS = F_2P$. This means that, with P as origin, we'd like to be able to show that

$$ABC + D_2E_2F_2 = QRS.$$

Again, these triangles are not similar, so MGT isn't applicable. But this equation has striking similarities with $(\bigstar)$. These similarities become more pronounced if we rewrite equation $(\bigstar)$ as

$$-2DEF + D_2E_2F_2 = -D_1E_1F_1 \qquad (\bigstar')$$

[7] The Nagel point is also (and less commonly) known as the **bisected perimeter point** [8] or the **splitting center** [7], since the cevians AD_2, etc., bisect the perimeter of the triangle, i.e. $AB + BD_2 = AC + CD_2 = s$.

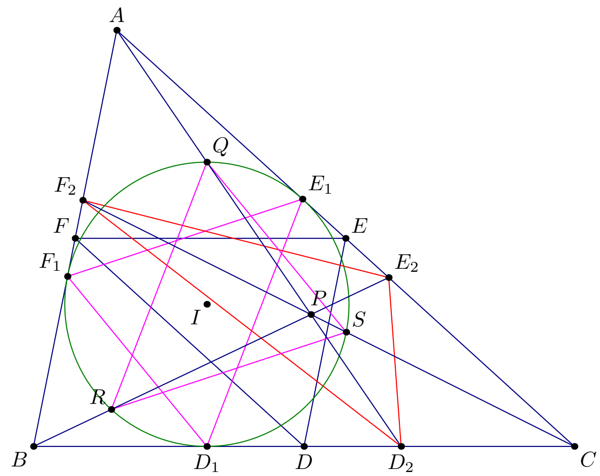

Figure 25: Problem 11

Triangle $-2DEF$ is simply a translation of ABC, and triangle $-D_1E_1F_1$ is a translation of QRS. The two equations are almost identical! This inspires the following observation:

Non-similar Figure Addition. *If three triangles (figures) satisfy $\omega_1\triangle_1 + \omega_2\triangle_2 = \triangle_3$, and if $\triangle_1'$ is a translated version of $\triangle_1$, then $\omega_1\triangle_1' + \omega_2\triangle_2$ is a translation of $\triangle_3$, regardless of the location of the origin.*

Indeed, if $\triangle_1' = \triangle_1 + \overrightarrow{\mathbf{V}}$,[8] then

$$\omega_1\triangle_1' + \omega_2\triangle_2 = \omega_1\triangle_1 + \omega_2\triangle_2 + \omega_1\overrightarrow{\mathbf{V}} = \triangle_3 + \omega_1\overrightarrow{\mathbf{V}}.$$

Now we can finish the problem. Beginning with equation ($\bigstar'$), translate $-2DEF$ to coincide with ABC. The above observation (with our origin still at P) proves that $Q'R'S' = ABC + D_2E_2F_2$ must be a translation of QRS. But since $A + D_2 = Q'$ lies on AD_2, and likewise for R' and S', triangles QRS and $Q'R'S'$ are also homothetic with center P. So the two triangles must be identical, and in particular, $Q = Q' = A + D_2$. So $\overrightarrow{\mathbf{AQ}} = \overrightarrow{\mathbf{PD_2}}$, proving the desired result. $\blacksquare$

[8]i.e. a translation by vector $\overrightarrow{\mathbf{V}}$

6 Additional Problems

Problem 12. Recall problem 4: "*Positively oriented equilateral triangles XAB, XCD, and XEF share a vertex X. If P, Q, and R are the midpoints of BC, DE, and FA respectively, prove that PQR is equilateral.*" Prove this problem using Napoleon's Theorem.

Problem 13 (Crux Mathematicorum)**.** A line parallel to the side AC of equilateral $\triangle ABC$ intersects BC at M and AB at P, thus making BMP equilateral as well. D is the center of BMP and E is the midpoint of CP. Determine the angles of ADE.

Problem 14. The following theorem appears in *Geometry Revisited* [5] as a special case of a theorem of Petersen and Schoute: *If ABC and $A'B'C'$ are two directly similar triangles, while $AA'A''$, $BB'B''$, $CC'C''$ are three directly similar triangles, then $\triangle A''B''C''$ is directly similar to $\triangle ABC$.*

a. Show that this theorem generalizes the triangle version of MGT.

b. What minor adjustment can be made to the statement of MGT to account for this generalization (and it's proof)?

c. Show that Napoleon's theorem is a special case of this theorem.

Problem 15 (Van Aubel's Theorem)**.** Given an arbitrary planar quadrilateral, place a square outwardly on each side, and connect the centers of opposite squares. Then these two lines perpendicular and of equal length.

Problem 16. Equilateral triangles AEB, BFC, CGD, DHA are erected outwardly on the sides of a plane quadrilateral $ABCD$.

a. Let M, N, O, and P be the midpoints of segments EG, HF, AC, and BD respectively. What is the shape of $PMON$?

b. M_d and M_a are the centroids of $\triangle DAH$ and $\triangle AEB$, and equilateral triangle M_dTM_a is oppositely oriented with respect to $ABCD$. Find the angles of triangle FTG.

c. M_a and M_c are the centroids of $\triangle AEB$ and $\triangle CGD$. Prove that segments M_aM_c and FH are perpendicular, and, in addition, $|FH| = \sqrt{3}\,|M_aM_c|$.

d. Equilateral triangles EWF, FXG, GYH, and HZE are oppositely oriented with respect to $ABCD$. Prove that quadrilaterals $ABCD$ and $WXYZ$ have the same area.

Problem 17. Let $\ell(P, QR)$ denote the line through point P perpendicular to line QR. Say that $\triangle XYZ$ *perpendicularizes* $\triangle ABC$ if $\ell(X, BC)$, $\ell(Y, CA)$, and $\ell(Z, AB)$ concur at a point. If $\triangle XYZ$ perpendicularizes $\triangle ABC$ and $\triangle DEF$, prove that $\triangle XYZ$ also perpendicularizes any linear combination of $\triangle ABC$ and $\triangle DEF$.

Problem 18 (Alex Zhai)**.**

a. In triangle ABC, AD, BE, and CF are altitudes. D_c and D_b are the projections of D onto AB and AC, respectively, and points E_a, E_c, F_a, and F_b are defined similarly. Prove that quadrilaterals BD_cD_bC, CE_aE_cA, and AF_bF_aB are cyclic.

b. Let O_A be the center of circle BD_cD_bC, and let T_a be the midpoint of altitude AD. Similarly define O_b, O_c, T_b, and T_c. If O is the circumcenter of triangle ABC, show that AOO_aT_a is a parallelogram, as well as BOO_bT_b and COO_cT_c.

c. Prove that lines O_aT_a, O_bT_b, and O_cT_c are concurrent.

Problem 19. Let $\mathcal{P}$ be a convex polygon in the plane. A real number is assigned to each point in the plane so that the sum of the number assigned to the vertices of any polygon similar to $\mathcal{P}$ is equal to 0. Prove that all the assigned numbers are equal to 0.

Problem 20. Show that I, G, and N (the incenter, centroid, and Nagel point of a triangle) are collinear in that order with $2 \cdot IG = GN$. *Hint: see problem 11.*

Problem 21. Given a convex quadrilateral $ABCD$, construct (with ruler and compass) a square of the same orientation with one vertex on each side of $ABCD$.

7 Solutions to Additional Problems

Problem 12. Recall problem 4: "*Positively oriented equilateral triangles XAB, XCD, and XEF share a vertex X. If P, Q, and R are the midpoints of BC, DE, and FA respectively, prove that PQR is equilateral.*" Prove this problem using Napoleon's Theorem.

Solution. In diagram 26, triangles BCJ, DEK, FAL are equilateral, and G, H, I, M, N, O are the centers of their respective triangles. By napoleon's theorem thrice, $\triangle HGM$, $\triangle IHN$, and $\triangle GIO$ are equilateral, and their centers T, U, V are also the centroids of $\triangle XBC$, $\triangle XDE$, and $\triangle XFA$ respectively (by problem 3). Again by napoleon's theorem (on the blue triangles), $\triangle TUV$ is equilateral, and the dilation ${}^{3/2}X$ carries $\triangle TUV$ to $\triangle PQR$.

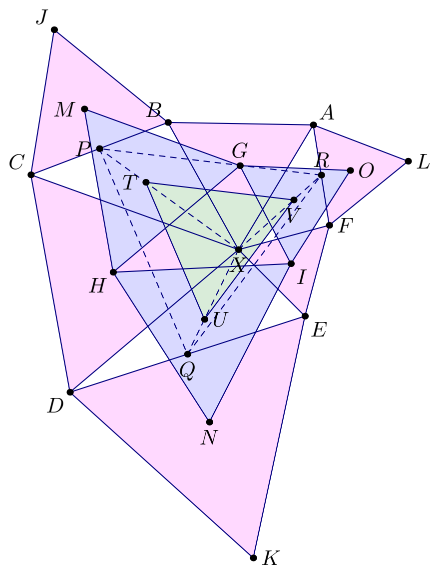

Figure 26: Problem 12

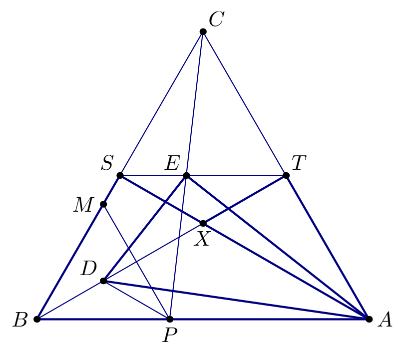

Figure 27: Problem 13

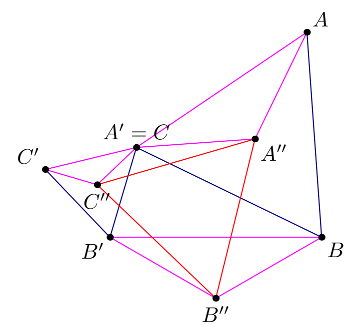

Figure 28: Problem 14.c

Problem 13 (Crux Mathematicorum)**.** A line parallel to the side AC of equilateral $\triangle ABC$ intersects BC at M and AB at P, thus making BMP equilateral as well. D is the center of BMP and E is the midpoint of CP. Determine the angles of ADE.

Solution. S and T are the midpoints of CB and CA, and X is the center of $\triangle ABC$. Let $BP/BA = k$. Because E must lie on line ST, and since $\triangle CST \sim \triangle CBA$, $SE/ST = BP/BA = k$. Also, since $\triangle BPD \sim \triangle BAX$, $BD/BX = BP/BA = k$. Thus, $(1-k)ABS + (k)AXT = ADE$, so $\triangle ADE$ is a $30-60-90$ triangle.

Problem 14. The following theorem appears in *Geometry Revisited* [5] as a special case of a theorem of Petersen and Schoute: *If ABC and $A'B'C'$ are two directly similar triangles, while $AA'A''$, $BB'B''$, $CC'C''$ are three directly similar triangles, then $\triangle A''B''C''$ is directly similar to $\triangle ABC$.*

a. Show that this theorem generalizes the triangle version of MGT.

b. What minor adjustment can be made to the statement of MGT to account for this generalization (and it's proof)?

c. Show that Napoleon's theorem is a special case of this theorem.

Solution. a. In the special case where A'' is on line AA', the similarity of degenerate triangles $AA'A''$, $BB'B''$, and $CC'C''$ simply means that $AA''/AA' = BB''/BB' = CC''/CC' = k$. Now, the fact that $A''B''C''$ is similar to

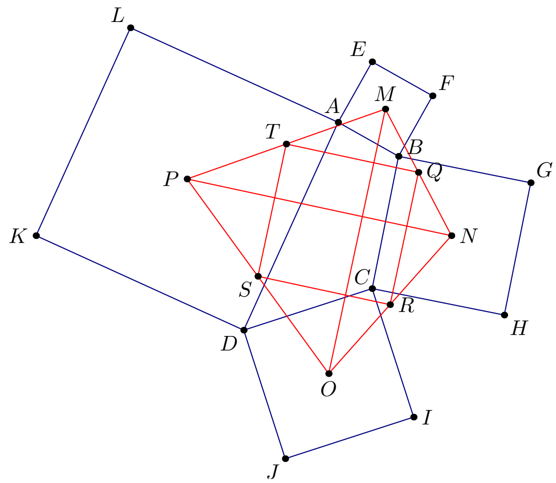

Figure 29: Van Aubel's Theorem

ABC is exactly the statement of MGT, since

$$(1-k)ABC + (k)A'B'C' = A''B''C''.$$

b. In the complex number proof of MGT (section 2.2), we required the two weights k and $1-k$ to be real numbers that add to 1. If we allow the weights to be *complex*, we obtain this generalization. The proof in section 2.2 remains unchanged.

c. The napoleon diagram has been relabeled in figure 28 to show the correspondence.

Problem 15 (Van Aubel's Theorem)**.** Given an arbitrary planar quadrilateral, place a square outwardly on each side, and connect the centers of opposite squares. Then these two lines perpendicular and of equal length.

Solution. Q, R, S, and T are the midpoints of the sides of quadrilateral $MNOP$. Since $\frac{1}{4}(A+F+G+C) = \frac{1}{2}(M+N) = Q$, and likewise around, averaging the four given squares proves that $QRST$ is itself a square:

$$\frac{1}{4}FBAE + \frac{1}{4}GHCB + \frac{1}{4}CIJD + \frac{1}{4}ADKL = QRST.$$

Since $\overrightarrow{\mathbf{MO}} = 2\overrightarrow{\mathbf{TS}}$ and $\overrightarrow{\mathbf{PN}} = 2\overrightarrow{\mathbf{TQ}}$, and since TS and TQ are equal in length and perpendicular, diagonals MO and NP must share this property, as desired.

Problem 16. Equilateral triangles AEB, BFC, CGD, DHA are erected outwardly on the sides of a plane quadrilateral $ABCD$.

a. Let M, N, O, and P be the midpoints of segments EG, HF, AC, and BD respectively. What is the shape of $PMON$?

b. M_d and M_a are the centroids of $\triangle DAH$ and $\triangle AEB$, and equilateral triangle M_dTM_a is oppositely oriented with respect to $ABCD$. Find the angles of triangle FTG.

c. M_a and M_c are the centroids of $\triangle AEB$ and $\triangle CGD$. Prove that segments M_aM_c and FH are perpendicular, and, in addition, $|FH| = \sqrt{3}\,|M_aM_c|$.

d. Equilateral triangles EWF, FXG, GYH, and HZE are oppositely oriented with respect to $ABCD$. Prove that quadrilaterals $ABCD$ and $WXYZ$ have the same area.

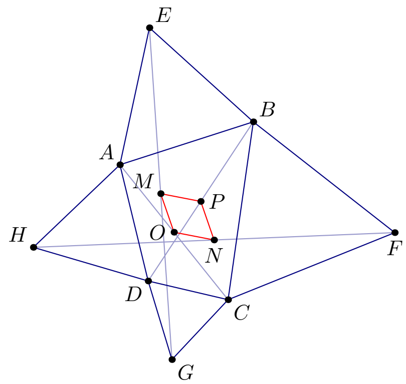

Figure 30: Problem 16.a

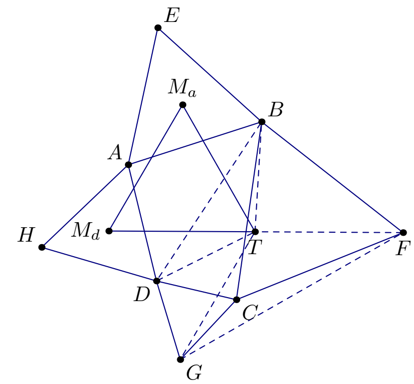

Figure 31: Problem 16.b

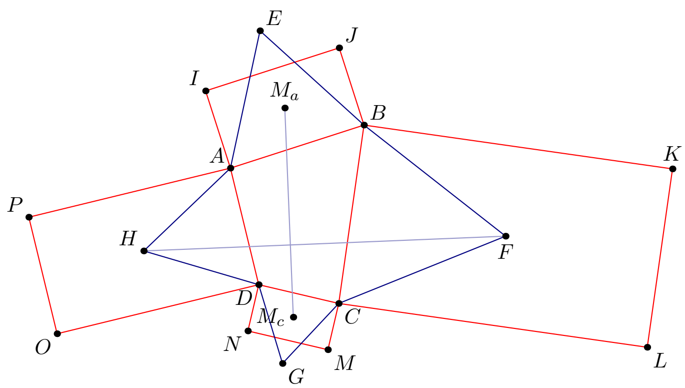

Figure 32: Problem 16.c

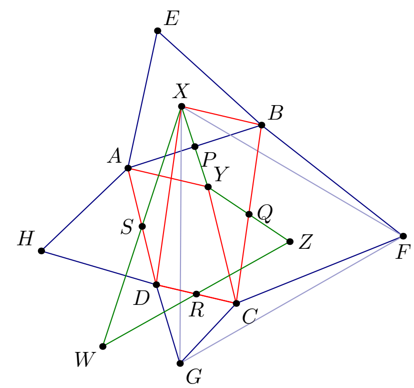

Figure 33: Problem 16.d

Solution. a. Since $\frac{1}{2}ABE + \frac{1}{2}CDG = OPM$ and $\frac{1}{2}BCF + \frac{1}{2}DAH = PON$, triangles $\triangle OPM$ and $\triangle PON$ are equilateral, so quadrilateral $MONP$ is a rhombus with vertex angles of $60°$ and $120°$.

b. By Napoleon's theorem on triangle DAB, T must be the center of the equilateral triangle erected on side DB, i.e. $\triangle DTB$ is a $30-120-30$ triangle. Now apply problem 9 to quadrilateral $DBFG$. Point C has $\angle GDC = \angle FBC = 60°$ and $\angle DGC = \angle BFC = 60°$, so problem 9 guarantees the existence of a point Y on the perpendicular bisectors of DB and FG so that $\angle DYB = \angle GYF = 120°$. T is on the perpendicular bisector of DB and has $\angle DTB = 120°$, so T is Y, and thus $\triangle GTF$ is a $30-120-30$ triangle.

c. This solution mimics the proof of Van Aubel's Theorem (problem 15). Erect rectangles $JBAI$, $KLCB$, $CMND$, and $ADOB$ with $\sqrt{3}:1$ side ratios as shown. The average of these four rectangles (with vertices in the listed order) is rectangle $WXYZ$ (not shown) which connects the midpoints of the sides of quadrilateral M_aFM_cH and which must be similar to the red rectangles. And as explained in problem 15, since the **Varignon Parallelogram** [5, Theorem 3.11] of quadrilateral M_aFM_cH has sides that are perpendicular with a ratio of $\sqrt{3}/1$, the diagonals FH and M_aM_c must have this property too.

d. Recall from problem 8 that, with C as center, $\triangle DGC + \triangle BCF = \triangle XGF$, proving that $D + B = X$ and so $XDCB$ is a parallelogram. Likewise, $ADCY$ is a parallelogram. Segments XB and AY are both parallel and congruent to DC, so $XAYB$ is a parallelogram, and AB and XY share the same midpoint P. The same goes for Q, R, and S. Thus, quadrilaterals $ABCD$ and $WXYZ$ have the same Varignon Parallelogram $PQRS$, so $\text{area}(ABCD) = 2 \cdot \text{area}(PQRS) = \text{area}(WXYZ)$.

Problem 17. Let $\ell(P, QR)$ denote the line through point P perpendicular to line QR. Say that $\triangle XYZ$ *perpendicularizes* $\triangle ABC$ if $\ell(X, BC)$, $\ell(Y, CA)$, and $\ell(Z, AB)$ concur at a point. If $\triangle XYZ$ perpendicularizes $\triangle ABC$ and $\triangle DEF$, prove that $\triangle XYZ$ also perpendicularizes any linear combination of $\triangle ABC$ and $\triangle DEF$.

Solution. The following lemma is key: *$\triangle XYZ$ perpendicularizes $\triangle ABC$ if and only if $\triangle ABC$ perpendicularizes $\triangle XYZ$.* To finish problem 17 with this property, note that if XYZ perpendicularizes ABC and DEF, then both ABC and DEF perpendicularize XYZ, say at points P and Q respectively. Define $GHI = \omega_1 ABC + \omega_2 DEF$ and $R = \omega_1 P + \omega_2 Q$. Segment GR – as a linear combination of AP and DQ – is perpendicular to YZ, and likewise for HR and IR. So GHI perpendicularizes XYZ at R, and the above lemma guarantees that XYZ must perpendicularize GHI.

We offer two very different proofs of the lemma.

Proof 1. We'll make use of the following property: lines PQ and RS are perpendicular if and only if $PR^2 + QS^2 = PS^2 + QR^2$. Indeed, using $\cdot$ as the vector dot product,

$$(\vec{\mathbf{P}} - \vec{\mathbf{R}}) \cdot (\vec{\mathbf{P}} - \vec{\mathbf{R}}) + (\vec{\mathbf{Q}} - \vec{\mathbf{S}}) \cdot (\vec{\mathbf{Q}} - \vec{\mathbf{S}}) = (\vec{\mathbf{P}} - \vec{\mathbf{S}}) \cdot (\vec{\mathbf{P}} - \vec{\mathbf{S}}) + (\vec{\mathbf{Q}} - \vec{\mathbf{R}}) \cdot (\vec{\mathbf{Q}} - \vec{\mathbf{R}})$$

is equivalent to $(\vec{\mathbf{P}} - \vec{\mathbf{Q}}) \cdot (\vec{\mathbf{R}} - \vec{\mathbf{S}}) = 0$ after expanding and simplifying.

For two triangles ABC and XYZ in the plane, let $\ell(Y, AC)$ and $\ell(Z, AB)$ meet at Q. $\triangle XYZ$ perpendicularizes $\triangle ABC$ if and only if $XQ \perp BC$, i.e. if and only if

$$\begin{aligned} 0 &= (CQ^2 - QB^2) + (BX^2 - XC^2) \\ &= (CQ^2 - QA^2) + (BX^2 - XC^2) + (AQ^2 - QB^2) \\ &= (CY^2 - YA^2) + (BX^2 - XC^2) + (AZ^2 - ZB^2). \end{aligned}$$

But this last expression makes no distinction between triangles ABC and XYZ, so if one triangle perpendicularizes another, the other must perpendicularize the first. ∎

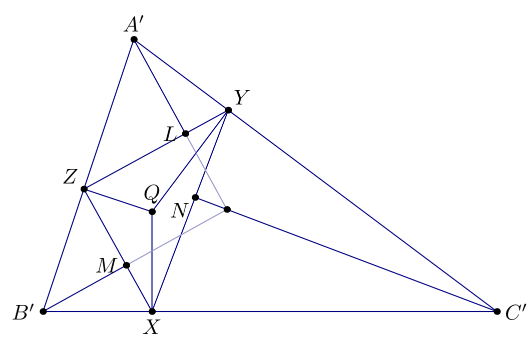

Figure 34: Proof 2 of perpendicularization lemma

Proof 2. Assume $\triangle XYZ$ perpendicularizes $\triangle ABC$ at point Q. Draw three lines: line ℓ_a through X parallel to BC, line ℓ_b through Y parallel to CA, and line ℓ_c through Z parallel to AB. These lines determine triangle $A'B'C'$ homothetic to $\triangle ABC$, and so ABC perpendicularizes XYZ if and only if $A'B'C'$ does.

Let L, M, N be the projections of A', B', C' onto YZ, ZX, XY respectively. Quadrilateral $A'ZQY$ is cyclic, so we may calculate

$$\angle C'A'L = 90^\circ - \angle ZYA' = 90^\circ - \angle ZQA' = \angle QA'B',$$

and likewise for $B'M$ and $C'N$. Thus,

$$\frac{\sin \angle C'A'L}{\sin \angle LA'B'} \cdot \frac{\sin \angle A'B'M}{\sin \angle MB'C'} \cdot \frac{\sin \angle B'C'N}{\sin \angle NC'A'} = \frac{\sin \angle QA'B'}{\sin \angle C'A'Q} \cdot \frac{\sin \angle QB'C'}{\sin \angle A'B'Q} \cdot \frac{\sin \angle QC'A'}{\sin \angle B'C'Q} = 1$$

i.e. $A'L$, $B'M$, and $C'N$ do in fact concur by the trigonometric form of Ceva's theorem. (They concur at the **isogonal conjugate** of Q with respect to triangle $A'B'C'$.) ∎

Problem 18 (Alex Zhai)**.**

a. In triangle ABC, AD, BE, and CF are altitudes. D_c and D_b are the projections of D onto AB and AC, respectively, and points E_a, E_c, F_a, and F_b are defined similarly. Prove that quadrilaterals BD_cD_bC, CE_aE_cA, and AF_bF_aB are cyclic.

b. Let O_A be the center of circle BD_cD_bC, and let T_a be the midpoint of altitude AD. Similarly define O_b, O_c, T_b, and T_c. If O is the circumcenter of triangle ABC, show that AOO_aT_a is a parallelogram, as well as BOO_bT_b and COO_cT_c.

c. Prove that lines O_aT_a, O_bT_b, and O_cT_c are concurrent.

Solution. a. By similar triangles AD_cD and ADB, $AD_c/AD = AD/AB$, i.e. $(AD_c)(AB) = (AD)^2$. Likewise, $(AD_b)(AC) = (AD)^2 = (AD_c)(AB)$, so BD_cD_bC is cyclic by the converse of power-of-a-point. Similar arguments work for the other two quadrilaterals.

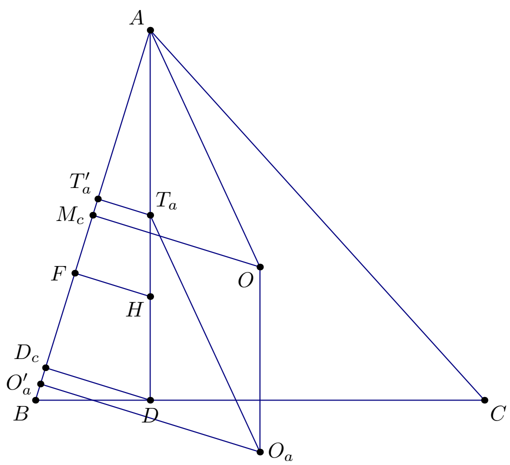

Figure 35: Problem 18.*b*

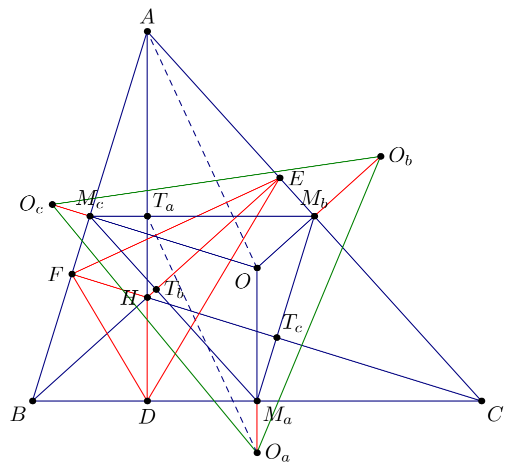

Figure 36: Problem 18.*c*

b. AT_a and OO_a are parallel because they are both perpendicular to BC, so it is only necessary to show they have the same length. Project T_a, O_a, and O onto AB at points T'_a, O'_a, and M_c, which must be the midpoints of AD_c, AB, and D_cB respectively. We have $T'_aA = D_cA/2$ and $O'_aM_c = BM_c - BO'_a = (BD_c + D_cA)/2 - (BD_c)/2 = D_cA/2$. Since the projections of T_aA and O_aO onto AB are equal in length, and since BA and AD are not perpendicular, $T_aA = O_aO$, as desired.

c. It can be calculated that $\angle OAC + \angle AEF = (90 - \angle CBA) + (\angle CBA) = 90$, so $AO \perp EF$, i.e. $O_aT_a \perp EF$. Thus, we wish to prove that $O_aO_bO_c$ perpendicularizes DEF (see problem 17).

Let the midpoints of BC, CA, AB be M_a, M_b, M_c, and let H be the orthocenter of ABC. Since $M_aM_bM_c$ is similar to ABC with half its size, and since O is the orthocenter of $M_aM_bM_c$, $OM_a = \frac{1}{2}AH$. Since we have already proved that $OO_a = \frac{1}{2}AD$, it follows that $M_aO_a = \frac{1}{2}HD$. So with H as origin, $O_aO_bO_c = M_aM_bM_c + \frac{1}{2}DEF$, i.e. $DEF = 2O_aO_bO_c - 2M_aM_bM_c$. And since $O_aO_bO_c$ perpendicularizes $M_aM_bM_c$ (at O) and $O_aO_bO_c$ (at its own orthocenter), triangle $O_aO_bO_c$ must perpendicularize the linear combination $2O_aO_bO_c - 2M_aM_bM_c = DEF$ by problem 17, as desired.

Problem 19. Let $\mathcal{P}$ be a convex polygon in the plane. A real number is assigned to each point in the plane so that the sum of the number assigned to the vertices of any polygon similar to $\mathcal{P}$ is equal to 0. Prove that all the assigned numbers are equal to 0.

Solution. Let $f(T)$ be the number associated with point T in the plane.

The idea is to use an extended version of problem 14 (illustrated for $n = 4$ in diagram 37), and then to let one of the polygons shrink to a single point X (figure 38).

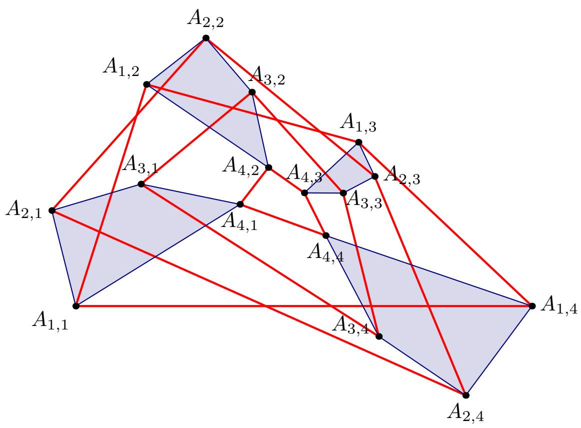

Figure 37: Generalization of problem 14

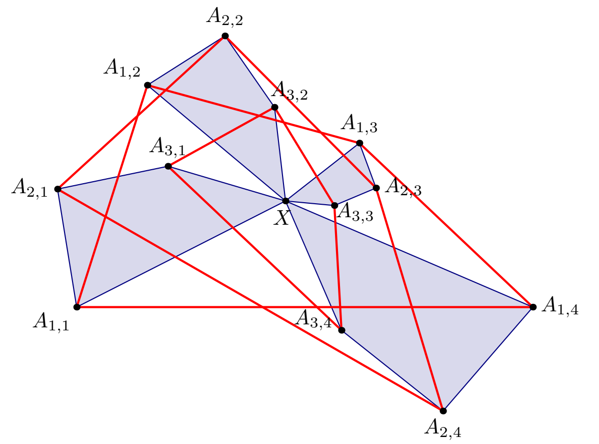

Figure 38: Problem 19

An arbitrary point X is chosen, polygon $\mathcal{R}_1 = A_{1,1}A_{1,2}\cdots A_{1,n}$ similar to $\mathcal{P}$ is drawn with no vertices at X, and then polygons $\mathcal{B}_i = A_{1,i}A_{2,i}\cdots A_{n-1,i}X$ $(1 \le i \le n)$ are drawn all similar to $\mathcal{P}$. It follows that each polygon $\mathcal{R}_i = A_{i,1}A_{i,2}\cdots A_{i,n}$ $(1 \le i \le n-1)$ is also similar to $\mathcal{P}$ (see problem 14). Thus, since $\sum_{b\in B_i} f(b) = 0$ and $\sum_{r\in R_j} f(r) = 0$ for any $1 \le i \le n$ and $1 \le j \le n-1$, we can add over polygons $\mathcal{B}_i$ and over polygons $\mathcal{R}_i$ to cancel out most of the terms:

$$0 = \sum_{i=1}^{n}\sum_{b\in\mathcal{B}_i} f(b) = n\cdot f(X) + \sum_{i=1}^{n}\sum_{j=1}^{n-1} f(A_{j,i}) = n\cdot f(X) + \sum_{j=1}^{n-1}\sum_{r\in\mathcal{R}_j} f(r) = n\cdot f(X) + 0.$$

This means $f(X) = 0$, and since point X was arbitrary, this holds for all points in the plane.

Problem 20. Show that I, G, and N (the incenter, centroid, and Nagel point of a triangle) are collinear in that order with $2\cdot IG = GN$. *Hint: see problem 11.*

Solution. Refer to figure 25 in problem 11. With origin P, define $D'E'F' = -2DEF$ and $D_1'E_1'F_1' = -D_1E_1F_1$. These are the triangles used – but never drawn – in equation ($\bigstar'$) on page 17, and they are translations of ABC and QRS respectively. Also set $X = 2D$. Since $\overrightarrow{\mathbf{AQ}} = \overrightarrow{\mathbf{PD_2}} = \overrightarrow{\mathbf{D_1X}} = \overrightarrow{\mathbf{D'Q'}}$, it follows that triangles $D'E'F'$ and $Q'R'S'$ are in the same relative position as ABC and QRS, i.e. $\overrightarrow{\mathbf{AD'}} = \overrightarrow{\mathbf{QD'_1}}$. And since $\overrightarrow{\mathbf{AD'}} = 3\overrightarrow{\mathbf{GP}}$ (since $ADD' \sim GDP$) and $\overrightarrow{\mathbf{QD'_1}} = 2\overrightarrow{\mathbf{IP}}$ (since $QD_1D_1' \sim ID_1P$), the conclusion follows.

Problem 21. Given a convex quadrilateral $ABCD$, construct (with ruler and compass) a square of the same orientation with one vertex on each side of $ABCD$.

Solution. First, we prove that it is impossible for $AB \perp CD$ and $AD \perp BC$ to occur simultaneously. Assume this does happen, label $AB \cap CD = X$ and $AD \cap BC = Y$, and assume (without loss of generality) that C is between X and D and between Y and B. Then

$$360^\circ > \angle B + \angle C + \angle D = 540^\circ - \angle C > 360^\circ,$$

contradiction. So it is safe to assume BC is not perpendicular to DA.

Given a point $P_i \in AB$, construct square $\square_i = P_iQ_iR_iS_i$ as follows: rotate line DA clockwise by 90° around P_i to intersect BC at Q_i (this intersection exists uniquely since $BC \not\perp DA$), and complete positively oriented square

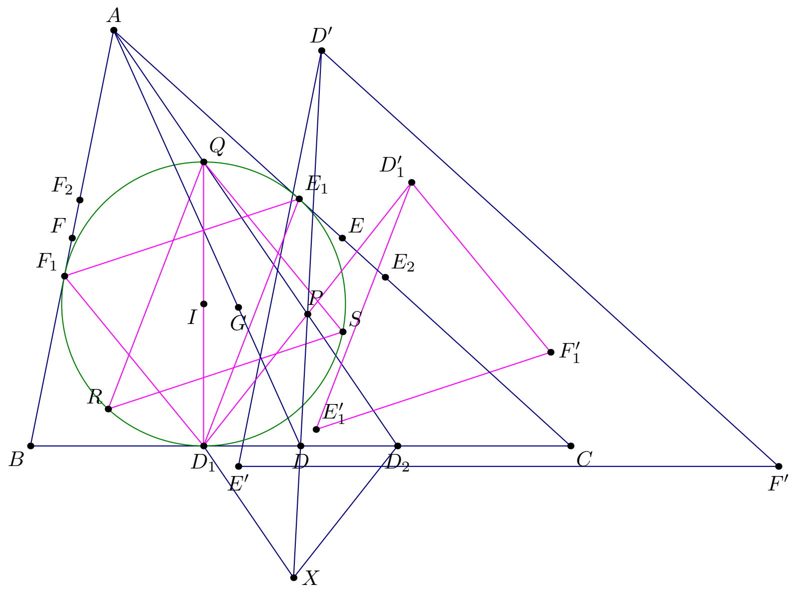

Figure 39: Problem 20

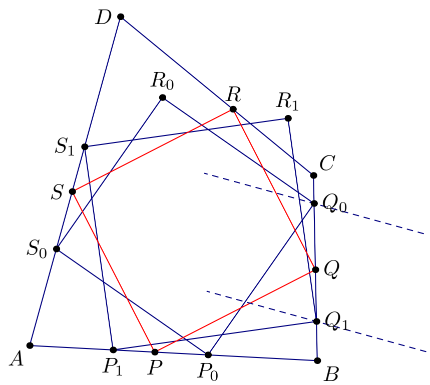

Figure 40: Problem 21

$P_iQ_iR_iS_i$. This square with vertex P_i has the properties that $Q_i \in BC$ and $S_i \in DA$ (by rotation), and furthermore, the construction proves that such a square is unique. Now it is only necessary to find the right $P \in AB$ so that the corresponding R lies on CD.

Choose two distinct points $P_0, P_1 \in AB$ and draw squares $\square_0$ and $\square_1$ as above. Choose any other point $P_t = (1-t)P_0 + (t)P_1$ on AB, and consider $\square'_t = P_tQ'_tR'_tS'_t = (1-t)\square_0 + (t)\square_1$. Since $Q'_t \in Q_0Q_1 \equiv BC$ and $S'_t \in S_0S_1 \equiv DA$, square $\square'_t$ satisfies the defining conditions for $\square_t$, i.e. $\square_t = \square'_t$. In particular, $R_t = R'_t = (1-t)R_0 + (t)R_1$, meaning R_t must lie on line R_0R_1. And since R_t covers all of this line as t varies, the locus of such points is exactly this line, i.e. $R = CD \cap R_0R_1$. The rest of the square can no be constructed now that the correct ratio $t = R_0R/R_0R_1$ is known.

There are a few exceptional cases to consider. If line R_0R_1 and line CD are identical, any point $P_t \in AB$ will produce a viable square $\square_t$. If the two lines are parallel but not equal, there is no square with the desired properties. And finally, if $R_0 = R_1$, then $R_t = R_0$ for all t, so all P_t work or no P_t work depending on whether or not R_0 is on line CD.

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