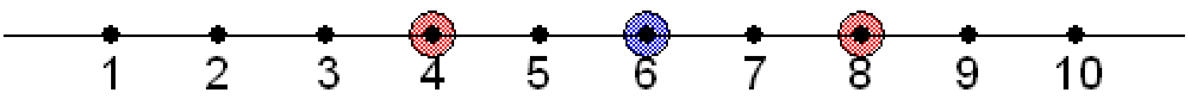
Mean Geometry

By Zachary Abel

Averages on the Number-Line

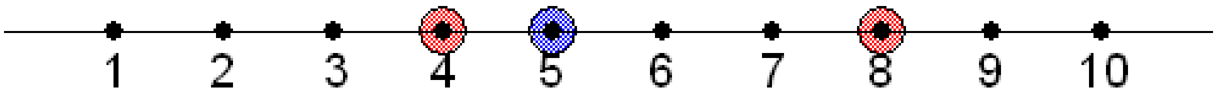
The average of 4 and 8 is

$$\frac{1}{2}(4) + \frac{1}{2}(8) = 6$$



A *weighted* average of 4 and 8, with *weights* of 3/4 and 1/4, is

$$\frac{3}{4}(4) + \frac{1}{4}(8) = 5$$

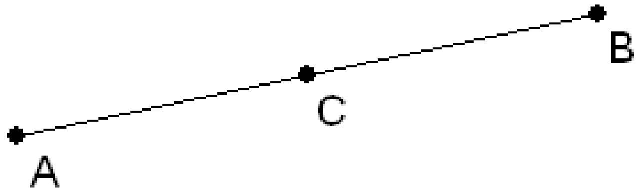


(the *weights* must add to 1)

Averages of Points

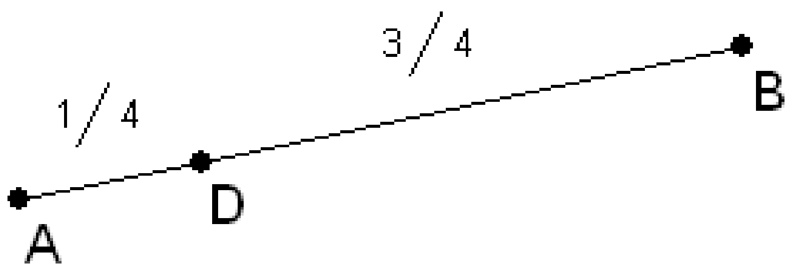
The average of two points A and B is the midpoint of segment AB, and is written as

$$\frac{1}{2}A + \frac{1}{2}B = C$$



The weighted average of two points A and B, for example $\frac{3}{4}A + \frac{1}{4}B$, is the point D that cuts segment AB into pieces that are $\frac{3}{4}$ and $\frac{1}{4}$ of segment AB:

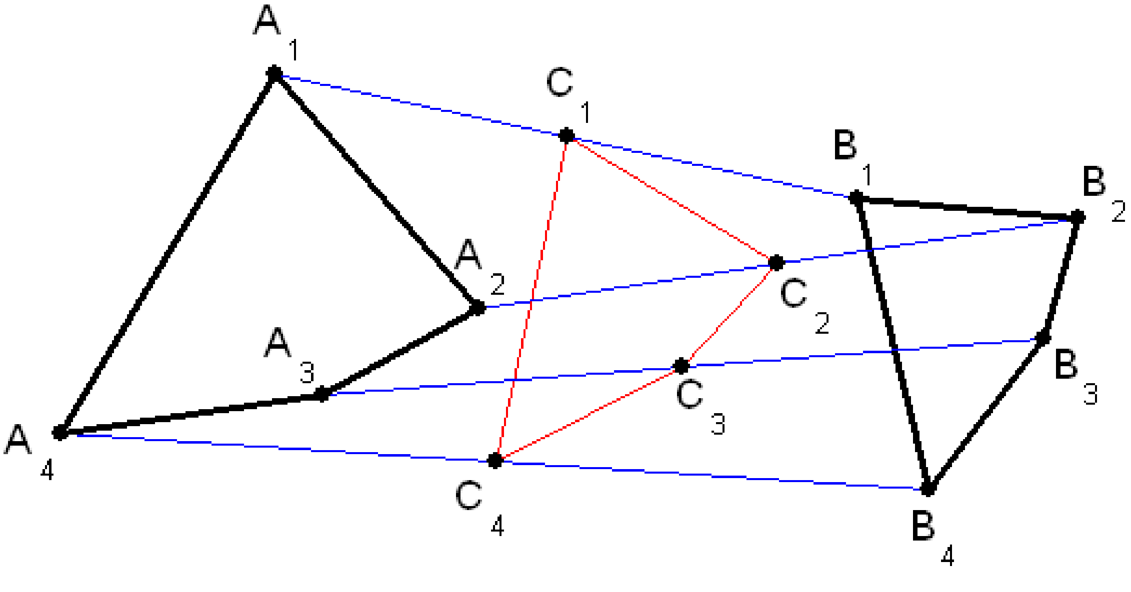
$$\frac{3}{4}A + \frac{1}{4}B = D$$



Averages of Figures

To take the average of two figures $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$, simply average each pair of corresponding points:

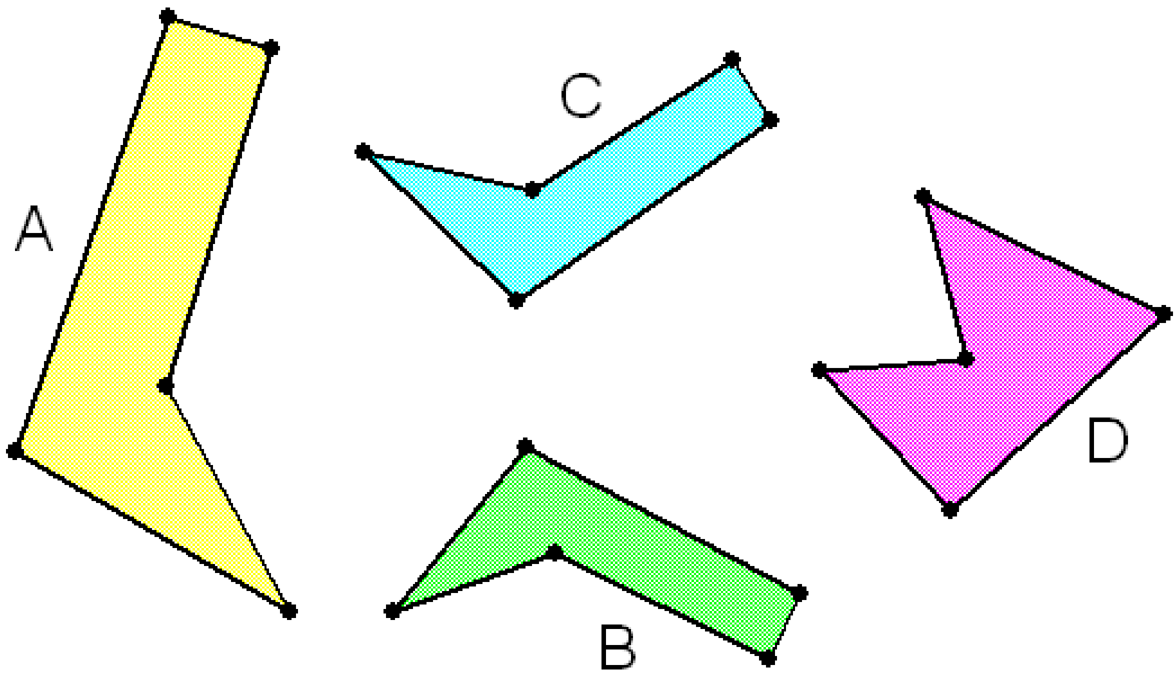
$$\frac{1}{2}A_1A_2A_3A_4 + \frac{1}{2}B_1B_2B_3B_4 = C_1C_2C_3C_4$$



Directly Similar

Two figures are **Similar** if they have the same shape, i.e. one is a scaled version of the other.

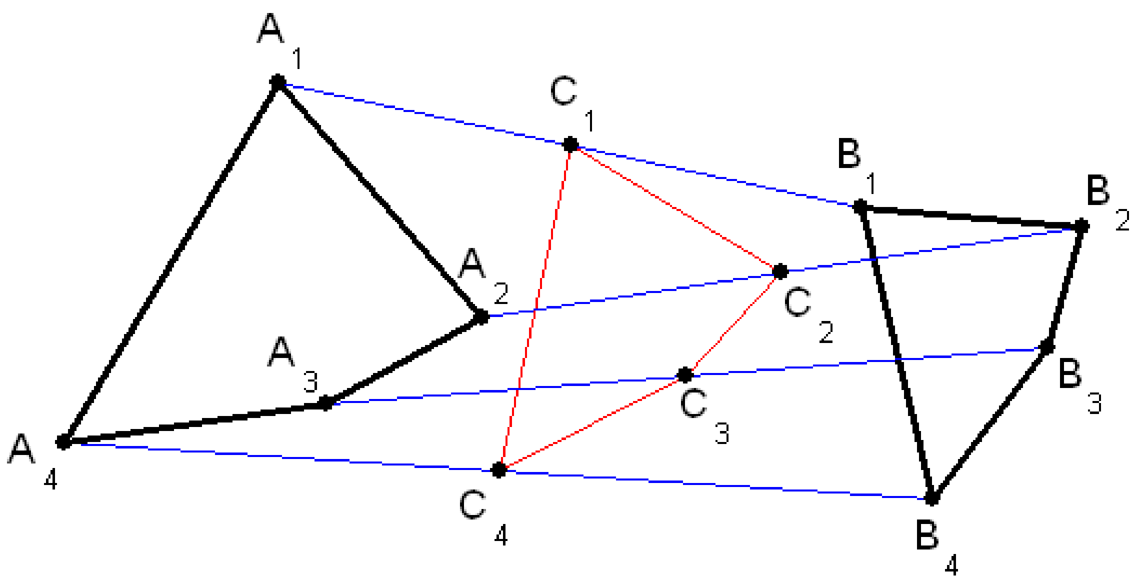
Two figures are **Directly Similar** if they also have the same orientation.



The Fundamental Theorem of Directly Similar Figures

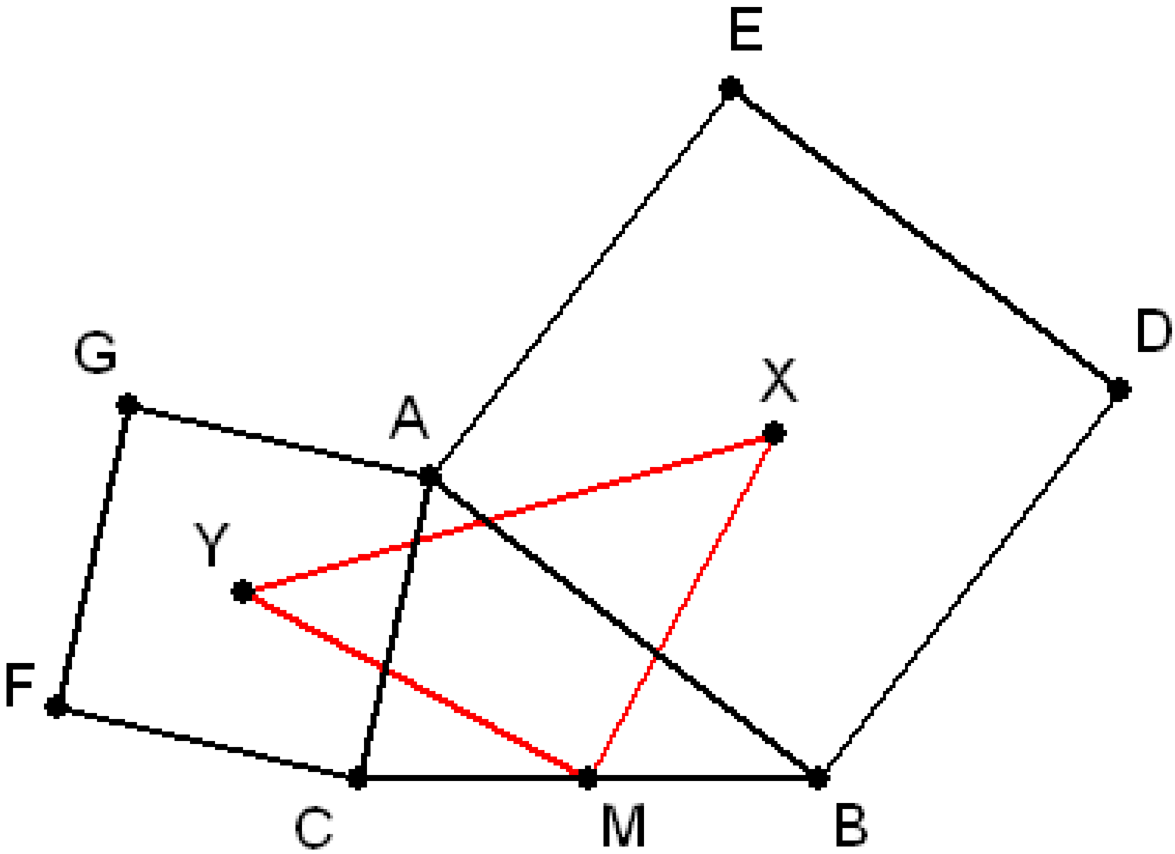
The weighted average of two directly similar figures forms a figure that is directly similar to the first two.

The average of two directly similar figures is another directly similar figure.



Right-Isosceles Triangle

On the outside of triangle ABC, erect two squares ABDE and ACFG, with centers at X and Y respectively. If M is the midpoint of side BC, prove that triangle XMY is a right-isosceles triangle.



What is $\frac{1}{2}DBA + \frac{1}{2}ACF$?

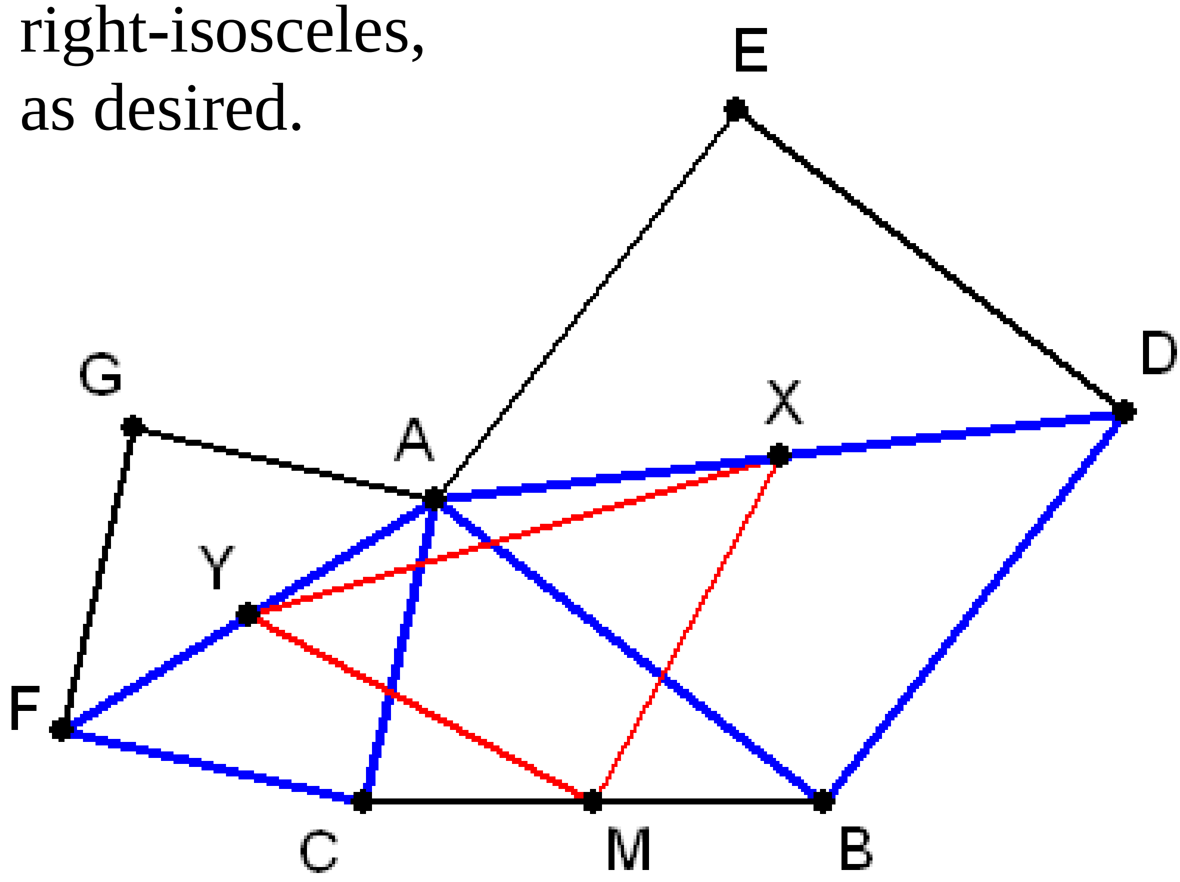
$$\frac{1}{2}D + \frac{1}{2}A = X$$

$$\frac{1}{2}B + \frac{1}{2}C = M$$

$$\frac{1}{2}A + \frac{1}{2}F = Y$$

$$\frac{1}{2}DBA + \frac{1}{2}ACF = XMY$$

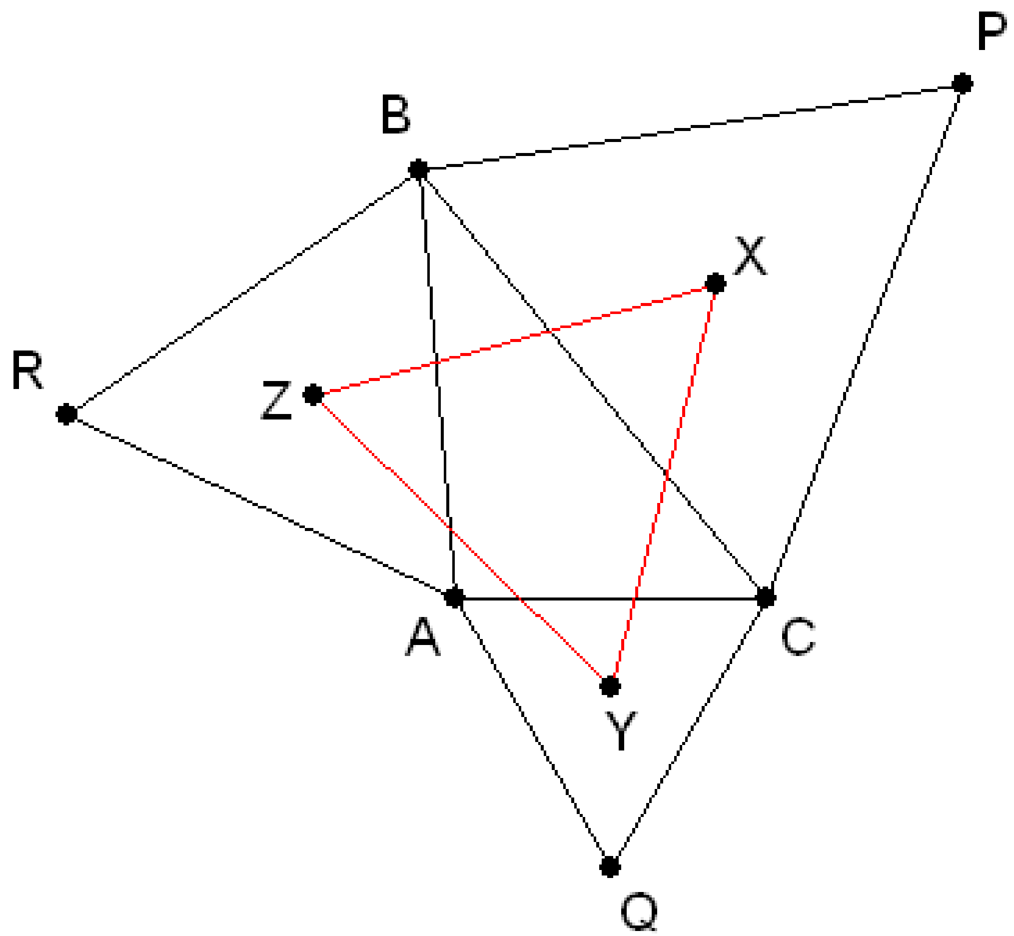
By the **Fundamental Theorem**, XMY is right-isosceles,
as desired.



Problem 2: Napoleon's Theorem

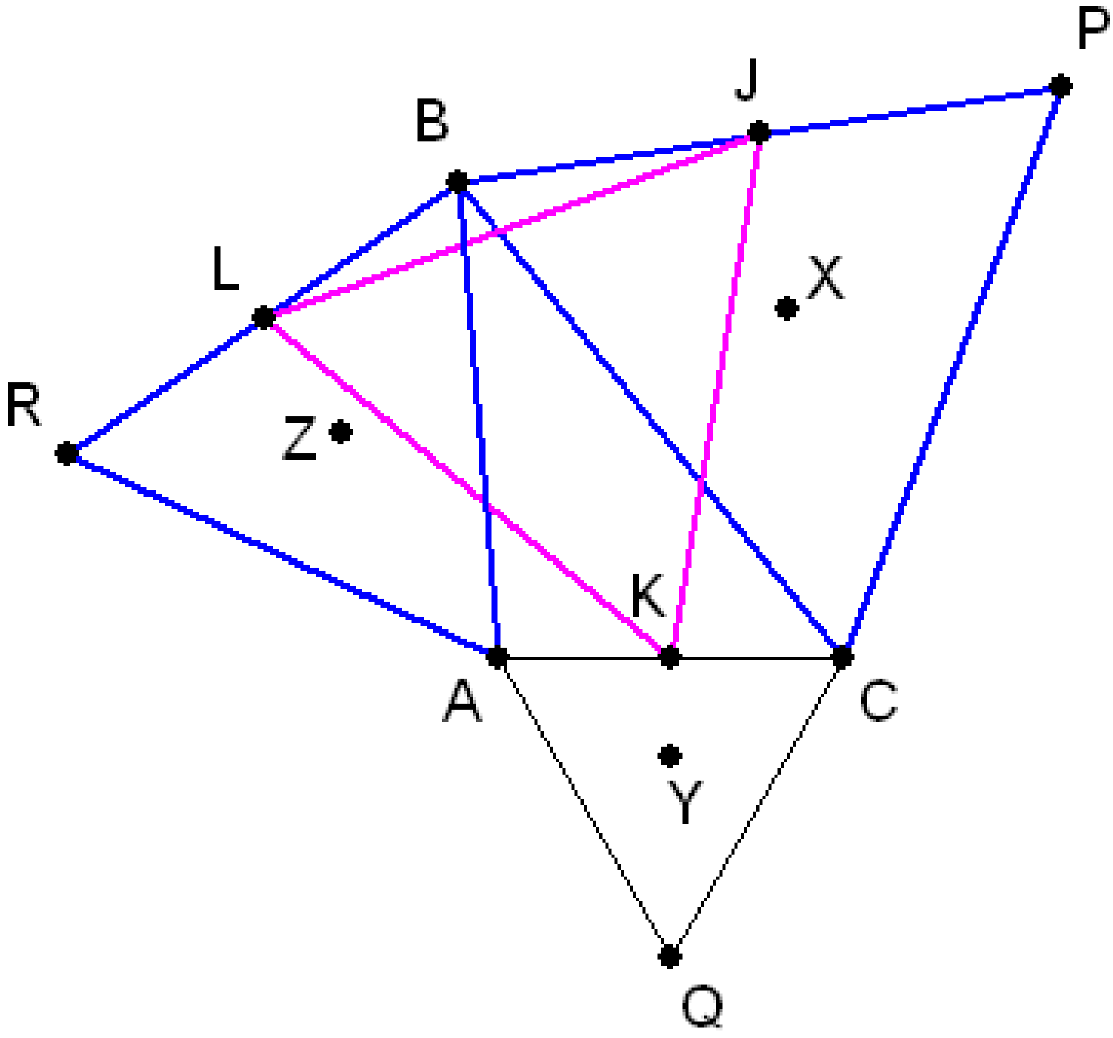
Napoleon Bonaparte

If Equilateral Triangles BCP, CAQ, and ABR are erected externally on the sides of any triangle ABC, their centers X, Y, and Z form an equilateral triangle.



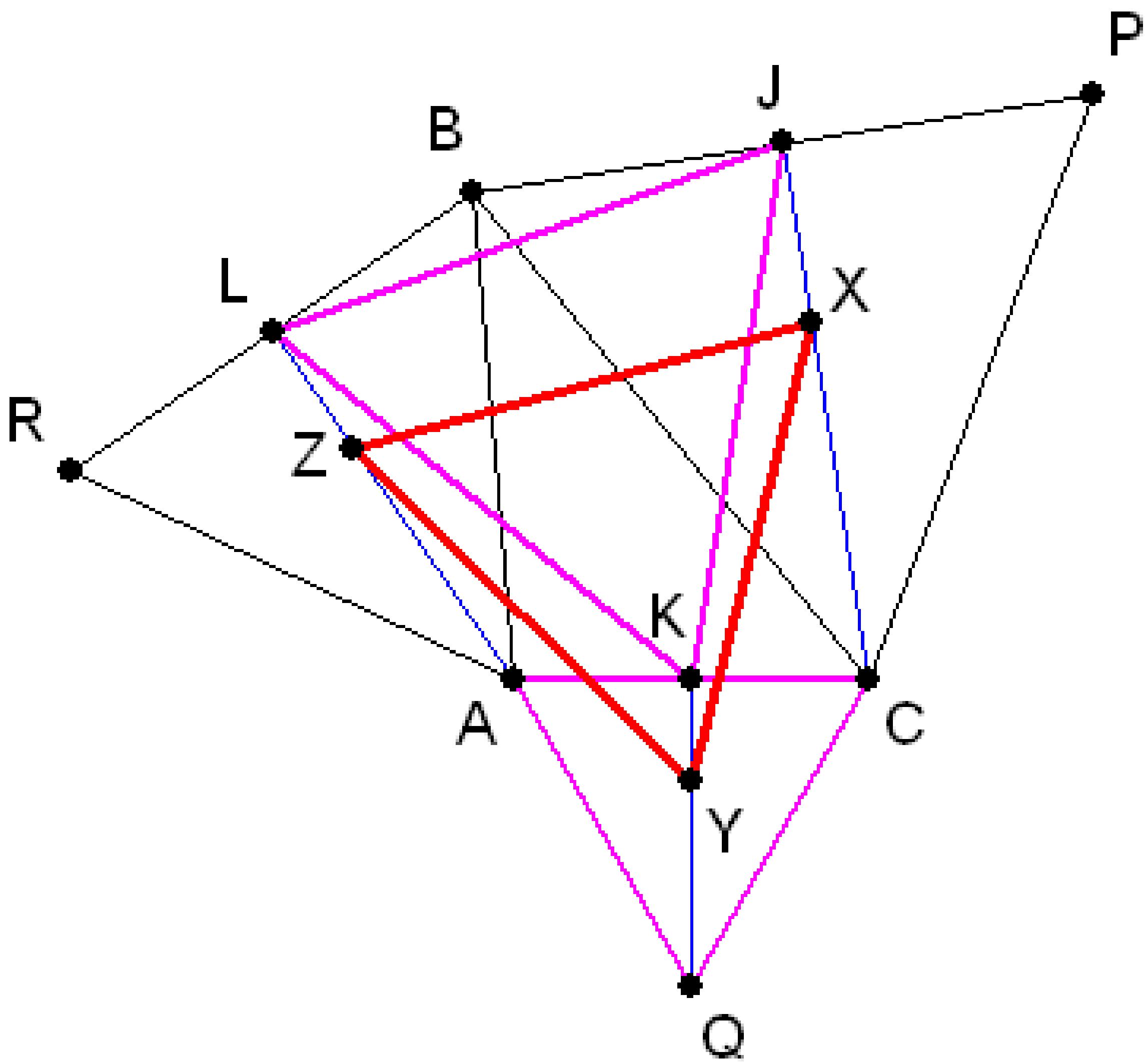
Proof of Napoleon's Theorem, Step 1

$$\frac{1}{2}\mathrm{PCB} + \frac{1}{2}\mathrm{BAR} = \mathrm{JKL}$$



Proof of Napoleon's Theorem, Step 2

$$\frac{2}{3}\text{JKL} + \frac{1}{3}\text{CQA} = \text{XYZ}$$



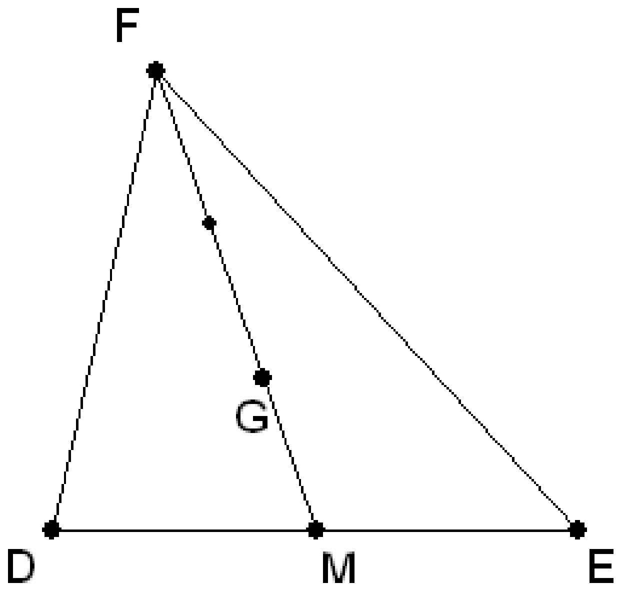
Averaging Three Figures: Medians and the Centroid

$$\frac{1}{3}D + \frac{1}{3}E + \frac{1}{3}F$$

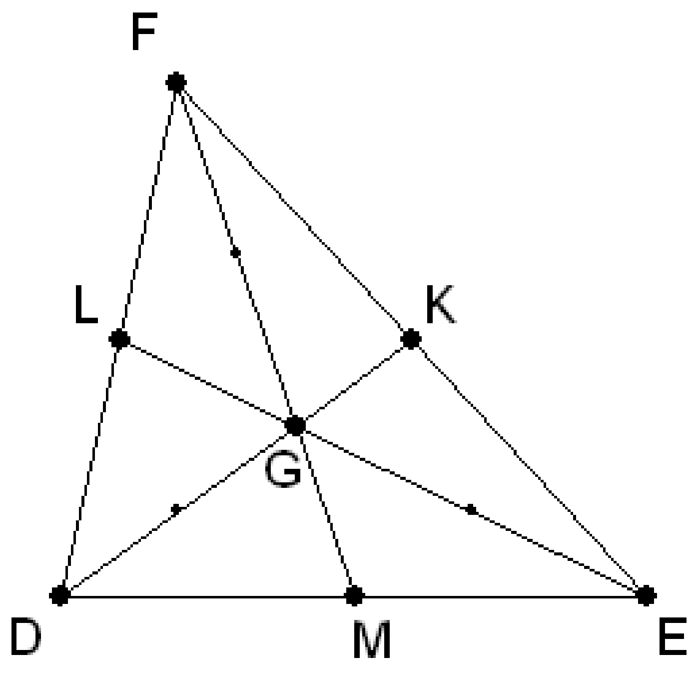
$$= \frac{2}{3}\left(\frac{1}{2}D + \frac{1}{2}E\right) + \frac{1}{3}F$$

$$= \frac{2}{3}M + \frac{1}{3}F$$

$$= G$$

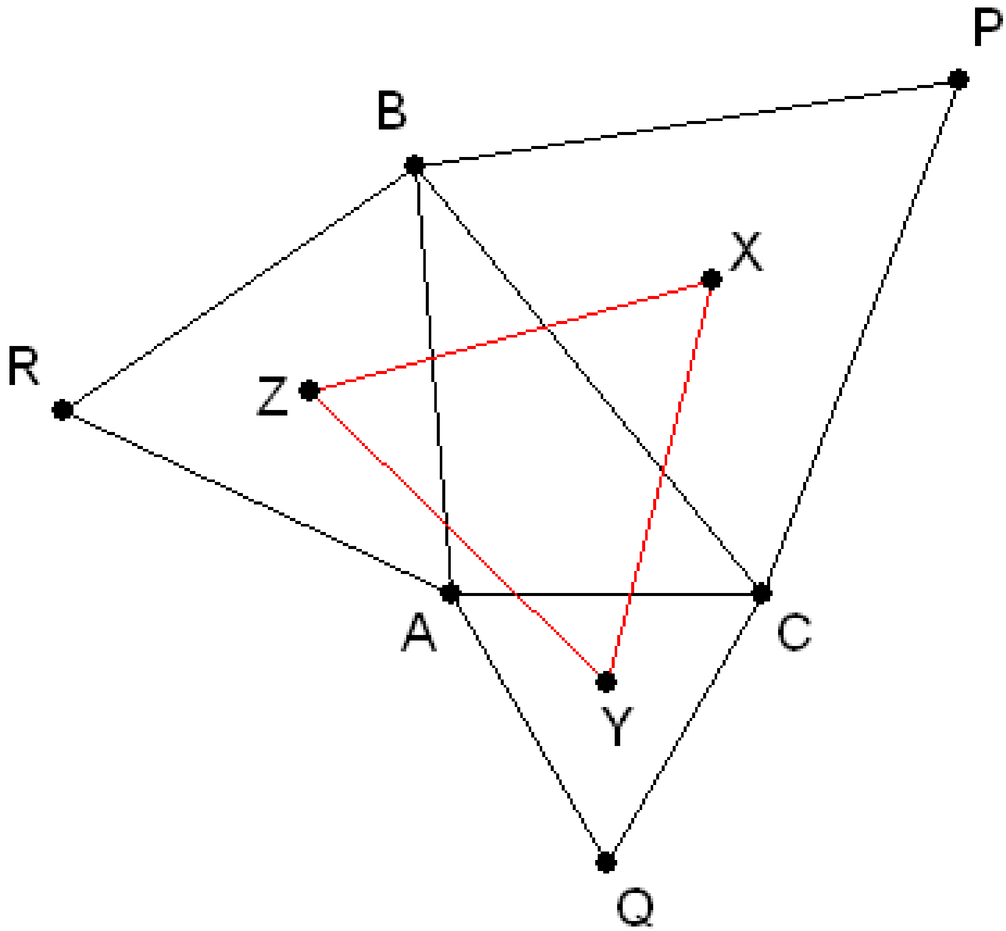


Averaging the three vertices of a triangle gives the centroid of that triangle.



One-Line Proof of Napoleon's Theorem

$$\frac{1}{3}\mathrm{PCB} + \frac{1}{3}\mathrm{CQA} + \frac{1}{3}\mathrm{BAR} = \mathrm{XYZ}$$

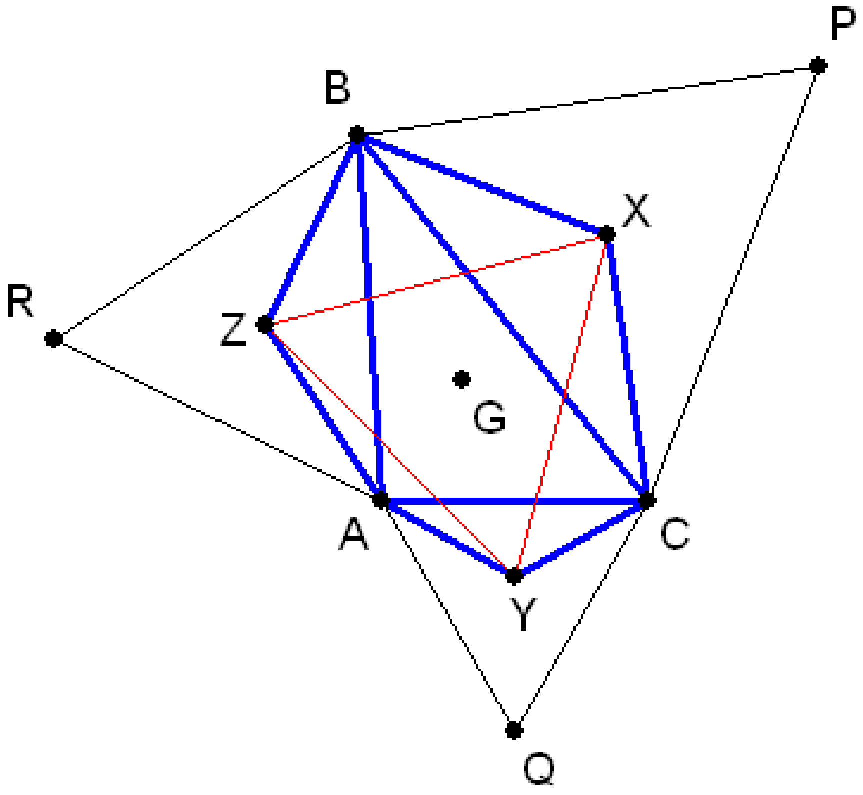


But Wait, There's More!

Show that the centroid of triangle ABC (point G) is the same as the centroid of XYZ (point H).

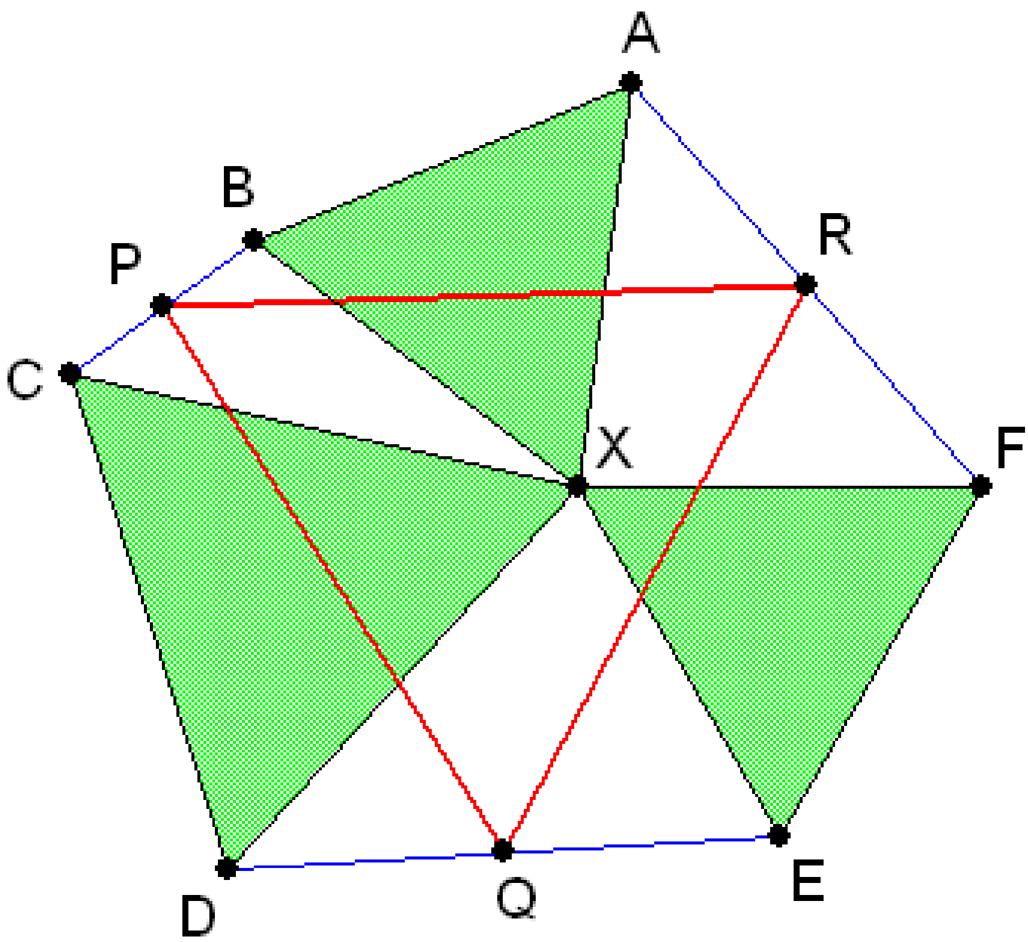
$$\frac{1}{3}\text{BCX} + \frac{1}{3}\text{CAY} + \frac{1}{3}\text{ABZ} = \text{GGH}$$

Triangle GGH is a 30-30-120 triangle, and since two of its vertices are the same, it is degenerate, i.e. G = H.



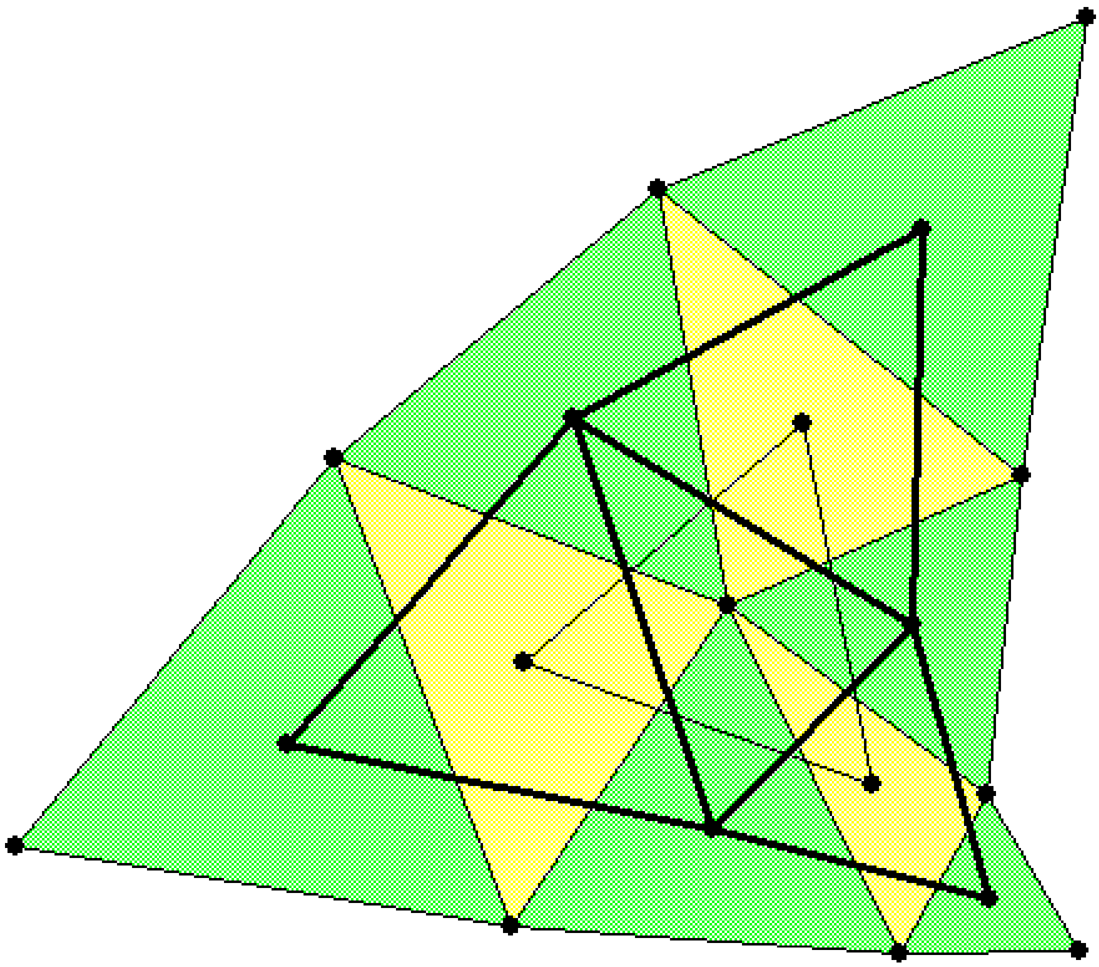
A Use for Napoleon's Theorem?

Three equilateral triangles XAB, XCD, and XEF share a common vertex X. If P, Q, and R are the midpoints of segments BC, DE, and FA respectively, prove that triangle PQR is equilateral.



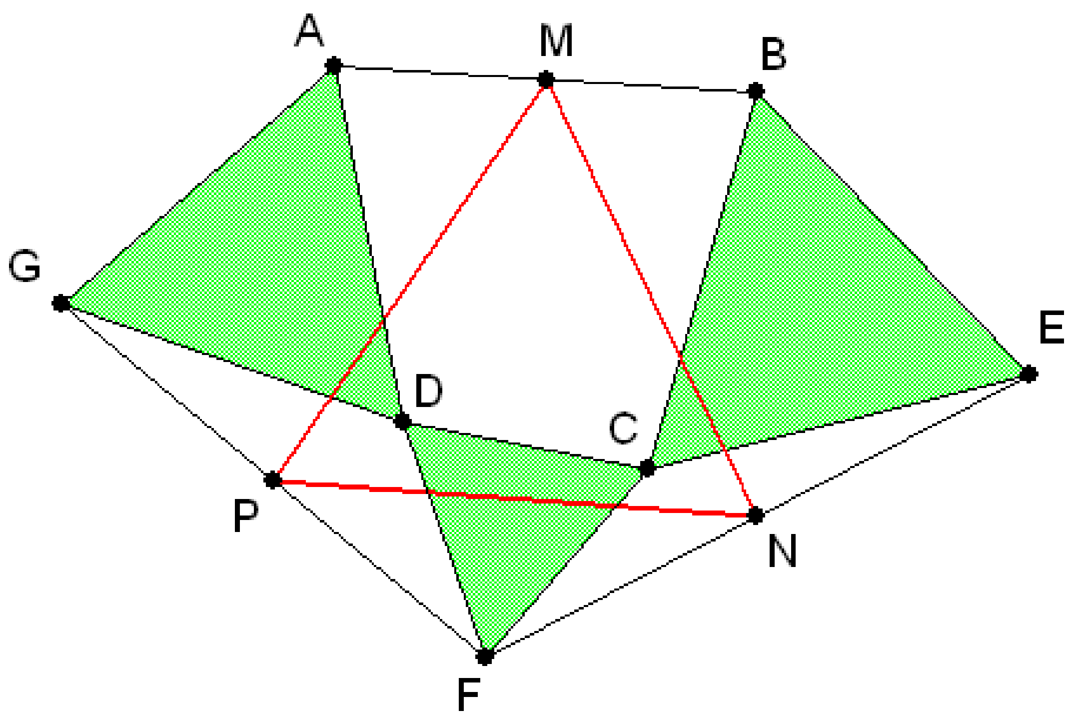
Proof by Napoleon and a Pretty Diagram

(Note the subtle Greenhill green-and-gold propaganda. Go Hornets!)



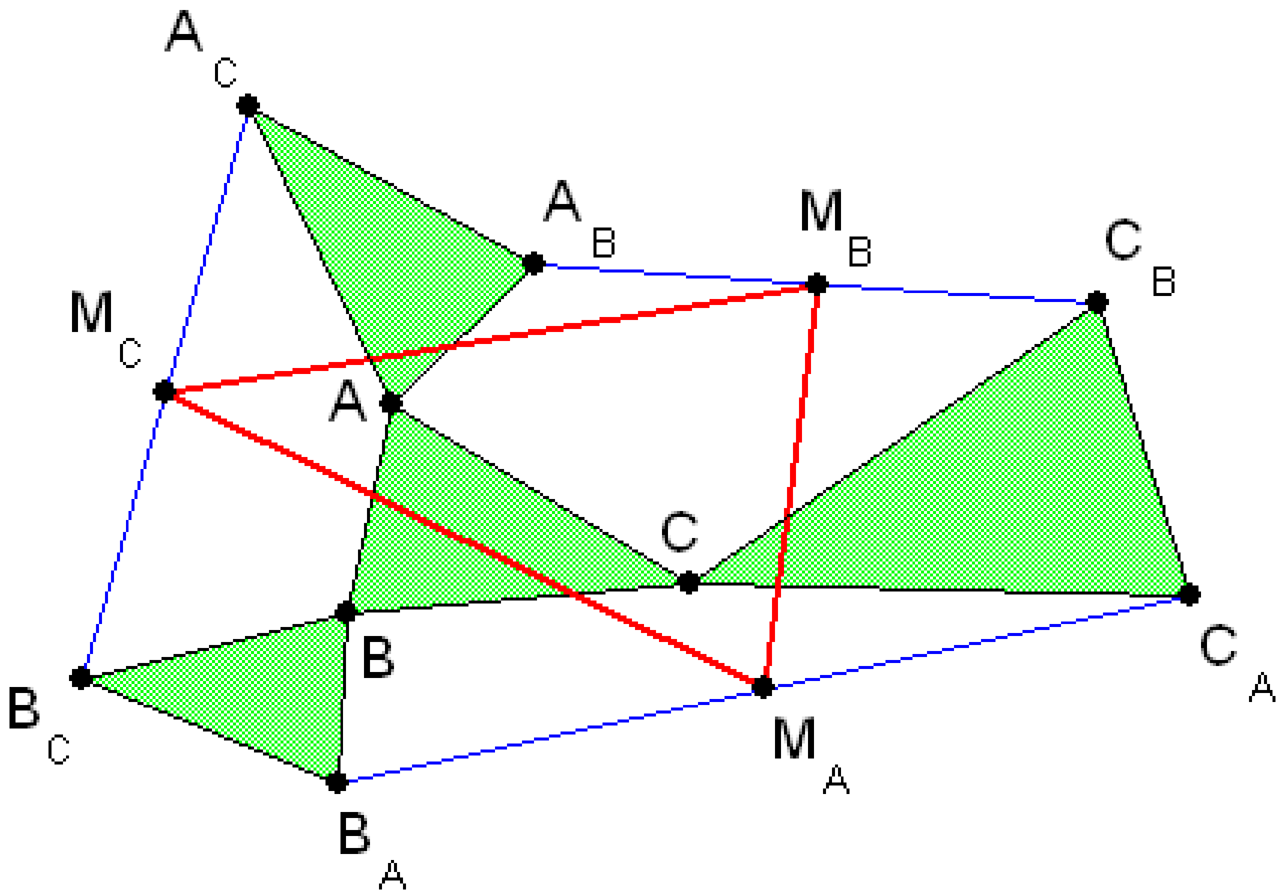
More Equilateral Triangles (Crux Mathematicorum)

In quadrilateral ABCD, M is the midpoint of AB. The three equilateral triangles BCE, CDF, and DAG are constructed on the outside of the quadrilateral. If N is the midpoint of EF and P is the midpoint of FG, prove that MNP is equilateral.



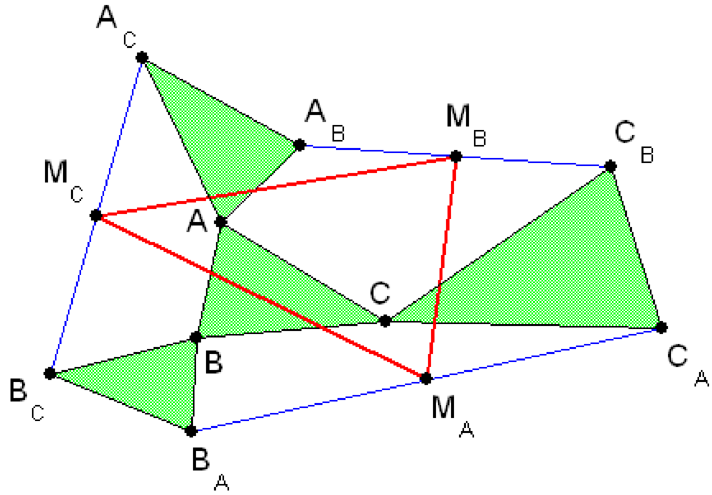
A Generalization: The Asymmetric Propeller

The four triangles ABC, AA_BA_C, B_ABB_C, and C_AC_BC are directly similar, and M_A, M_B, and M_C are the midpoints of B_AC_A, C_BA_B, and A_CB_C. Show that $M_AM_BM_C$ is also similar to ABC.

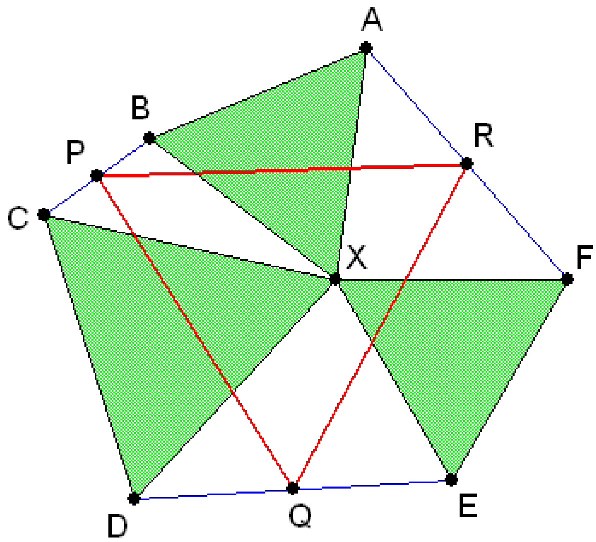


Why is it a Generalization?

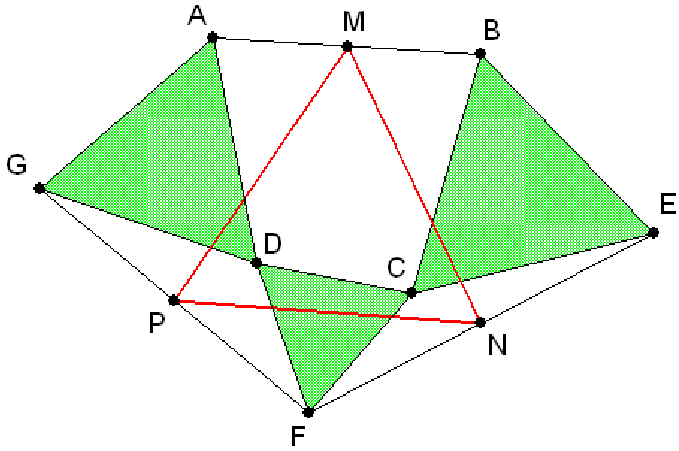
Asymmetric Propeller:



First Problem

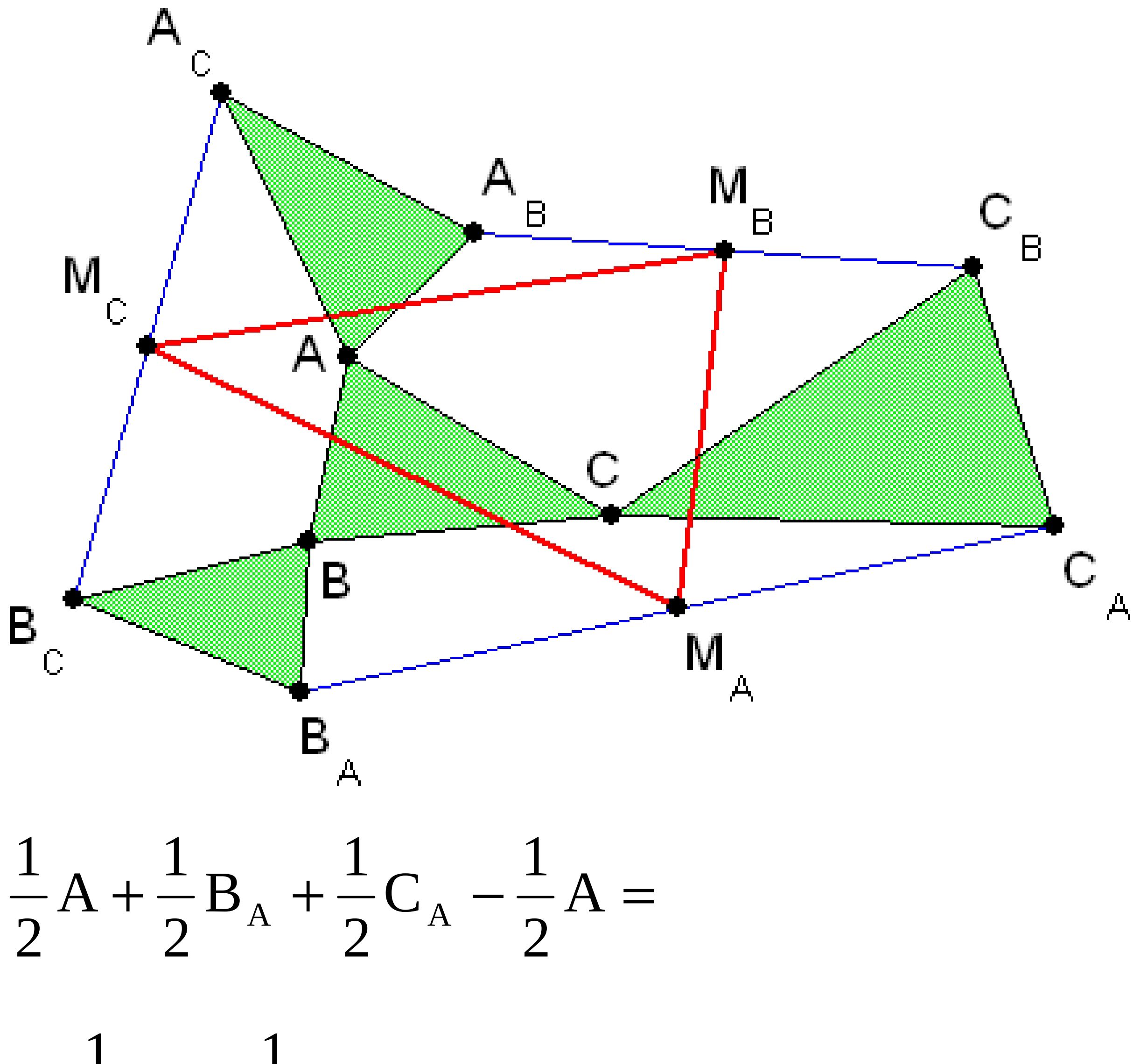


Second Problem



Proof of Propeller

$$\frac{1}{2}AA_BA_C + \frac{1}{2}B_ABB_C + \frac{1}{2}C_AC_BC - \frac{1}{2}ABC =$$
$$= M_AM_BM_C$$



$$\frac{1}{2}A + \frac{1}{2}B_A + \frac{1}{2}C_A - \frac{1}{2}A =$$

$$= \frac{1}{2}B_A + \frac{1}{2}C_A = M_A$$