Pushing Hypercubes Around

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I. Introduction

A. Background

We study collections of identical, connected modules which may relocate relative to each other; such a collection is called a *modular metamorphic system* (see [CPE], [MKK], [PEC], [RV], [YMK$^+$]). While such a system must always remain connected, it is possible for such a system to reconfigure its shape through successive motions of individual modules, either by rotation and sliding (e.g., [MKK]) or by expansion and contraction (e.g., [PEC]).

The theory of modular metamorphic systems has important applications in the study and use of *reconfigurable robots*, small, modular robots with limited motion abilities (see [Yi], [Ch], [MKK]). Such robots reorganize themselves by changing shape locally, while maintaining connectivity. Reconfigurable robots are easily adaptable, as they are relatively inexpensive to produce and exhibit high fault tolerance.

Given the desire to have reconfigurable robots take on specific configurations, it is natural to ask whether a collection of modules can achieve specified configurations. Formally, the *motion planning problem* for a modular metamorphic system asks for a sequence of motions which transform a given configuration of modules V into a specified configuration V'. We denote the motion planning problem asking for a transformation taking V to V' by $[V \mapsto V']$. When a solution to $[V \mapsto V']$ exists, we say that $[V \mapsto V']$ is *feasible*.

For configurations V and V' of two-dimensional, hexagonal modules, the problem $[V \mapsto V']$ is feasible whenever the configurations have the same number of modules and do not contain a single three-module pattern, as shown by Nguyen, Guibas, and Yim [NGY]. Recently, Dumitrescu and Pach [DP] showed that the motion planning problem is even simpler for square modules in two dimensions. Indeed, for any two configurations V and V' of n square modules, the problem $[V \mapsto V']$ is feasible (see [DP]).

We find a similar reconfiguration result for metamorphic systems of d-dimensional hypercubic modules. In particular, we will show in Section II-B that for any two n-module configurations V and V' of d-dimensional hypercubic modules, the problem $[V \mapsto V']$ is feasible. This result fully generalizes Dumitrescu and Pach's [DP] result for squares. Furthermore, our result for $d = 3$ affirmatively answers the "Pushing Cubes Around" problem proposed by O'Rourke at CCCG 2007 [DO].

B. Preliminaries

1) The Setting: Following the structure of [DP], we consider d-dimensional space with orthonormal basis $\{x_1, \ldots, x_n\}$ partitioned into a integer grid $\mathcal{G}$ of d-cubic cells. Each such cell may be empty or may be occupied by a module. For clarity, let *face* and *edge* denote $(d-1)$-dimensional and $(d-2)$-dimensional facets of a module, respectively. We say that two modules in this grid are *face-adjacent* if the Euclidean distance between their centers is exactly a unit, i.e. they have a common $(d-1)$-dimensional face. Similarly, two modules are *edge-adjacent* if they share an edge but not a face.

An n-module system V is said to be *connected* when the induced graph $G(V)$ is connected, where $G(V)$ is the graph having as its vertex set the modules of V and edge-set the pairs of face-adjacent modules of V.

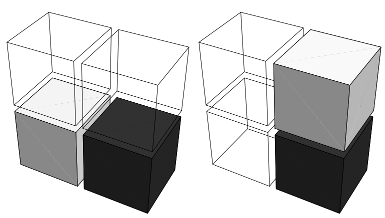
The configuration V partitions $\mathcal{G} \setminus V$ into a number of disjoint, face-connected components, exactly one of which is infinite. Let the *outer boundary* of V be the collection of module faces adjacent to this infinite region and let $B_{\mathrm{out}}(V)$ denote the set of modules in V which have at least one face on the outer boundary.

2) Reconfiguration of Modules: In this paper, we are concerned with modular metamorphic systems in which modules reconfigure by *rotation* and *sliding*, as illustrated in Fig. 1. These moves are the d-dimensional analogs of those in the *rectangular model* studied in [DP]:

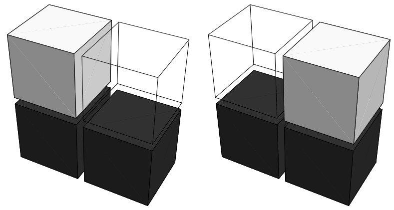
- *Rotation:* If a module b has two adjacent faces f and f' such that module a is adjacent to b at f,

the grid cell adjacent to b at f' is empty, and the grid cell edge-adjacent to b along $f \cap f'$ is also empty, then a may move to the cell adjacent to f'. (Fig. 1(a))
- *Sliding:* If two modules b and b' are adjacent, and a module a is adjacent to both b and an empty cell adjacent to b', then a may move to this empty cell. (Fig. 1(b))

In other words, we allow the movements described in [DP] to occur in any 2-dimensional plane.



(a) *Rotating*: the white module rotates around the black one.



(b) *Sliding*: the white module slides across the two black ones.

Fig. 1: Illustrations of the two legal moves in dimension $d = 3$. In both cases, the cells drawn with only outlines must be empty.

A *reconfiguration* of an n-module system V is a sequence of n-module configurations $\{V_t\}_{t=0}^{t_{\text{end}}}$, such that each V_t is connected and such that V_t can be obtained from V_{t-1} via a sequence of rotations and slides. In this paper we restrict our attention to *sequential reconfigurations* (as opposed to *parallel reconfigurations*), that is, reconfigurations in which V_t and V_{t+1} differ only by a single move.

II. Main Theorem

In this section we prove our main result:

Theorem 1. *Given any two connected configurations V and V' each having $n \geq 2$ modules, there exists a reconfiguration of V into V', i.e. the problem $[V \mapsto V']$ is feasible with only rotations and slides.*

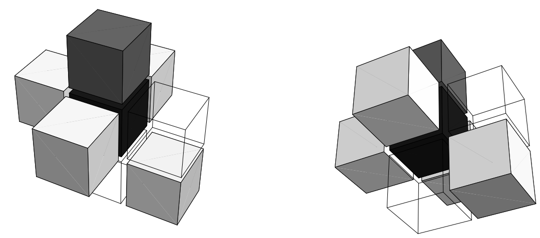


Fig. 2: Lemma 4: removing x (black) disconnects y (gray) from the boundary $B_{\text{out}}(V)$. (Two views are presented.)

Theorem 1 fully generalizes the two-dimensional result of [DP] into arbitrary dimensions. Our method generalizes and simplifies the approach of [DP].

A. Preliminary Reduction

As in [DP], we prove our main result by showing that any configuration can be reconfigured into a straight chain of modules. This suffices to prove the result, as it follows that any configuration V may be reconfigured into this canonical straight position, which may then be reconfigured into any other position V'. (Note that the straight configuration may easily be relocated in space by simple moves.)

B. Proof of Main Theorem

The proof of the main theorem will follow from a series of Lemmata. We will then give an iterative algorithm to reconfigure any connected configuration V into a straight chain.

Definition 2. *A module m in a connected configuration V is said to be an* articulation module *(or simply* articulate*) if it corresponds to an articulation node in $G(V)$, the connectivity graph of V. That is, if $V \setminus \{m\}$ is disconnected.*

Lemma 3. *Any connected graph G on $n \geq 2$ vertices contains at least 2 distinct non-articulate nodes.*

Proof: As G is connected, we may find a spanning tree $T \subset G$. Any leaf of T must be a non-articulation point of G, as its removal leaves the rest of T, and hence the rest of G, connected. It is well-known that any tree on at least 2 vertices has at least 2 leaves, so we are done. ■

Lemma 4. *Suppose $x \in B_{\text{out}}(V)$ is an articulate module, and that x is adjacent to a module y (along face f of x) such that the connected component of $V \setminus \{x\}$ containing y is disjoint from $B_{\text{out}}(V)$. Then:*

(i) *The face f^{op} of x opposite f is on the boundary of V, and*
(ii) *any module $w \neq y$ adjacent to x is in a component of $V \setminus \{x\}$ not disjoint from $B_{\text{out}}(V)$,*
(iii) *x is adjacent to at least one such module $w \neq y$.*

Proof: Let g be a face of x edge-adjacent to f, and suppose that g is on the boundary of V. Let p be the empty cell adjacent to x at g, and let q be the cell not containing x adjacent to both p and y. Since g is on the boundary, p is empty. But since y is not in $B_{\text{out}}(V)$, q must contain a module m_q. However, this means y is adjacent to m_q, and $m_q \in B_{\text{out}}(V)$, a contradiction. Thus, the only face of x that could be on the boundary of V is f^{op}. Finally, since x is in $B_{\text{out}}(V)$, this face must indeed be on the boundary, proving part (i).

Now suppose $w \neq y$ is adjacent to x along face h. Let r be the cell adjacent to x at f^{op}, let s be the cell containing w, and let t be the cell adjacent to r and s not containing x. If t is empty, then clearly $w \in B_{\text{out}}(V)$. Otherwise, the module m_t in cell t is adjacent to r (which is empty), so $m_t \in B_{\text{out}}(V)$. And since w is adjacent to m_t, we have proven part (ii).

Finally, since x is an articulate point of V, x has degree at least 2, so it is adjacent to at least one module $w \neq y$, proving part (iii). ■

Definition 5. *For a configuration V of n modules, a module m on $B_{\text{out}}(V)$ is called a* nearly non-articulate module *if $V \setminus \{m\}$ has exactly two connected components, one of which is disjoint from $B_{\text{out}}(V)$.*

Lemma 6. *For any configuration V of size $n \geq 2$ and a module $s \in B_{\text{out}}(V)$, there is either a non-articulate module or a nearly non-articulate module of V in $B_{\text{out}}(V) \setminus \{s\}$.*

Proof: By Lemma 3, V contains two non-articulate modules, and hence V has at least one non-articulate module $m_1 \neq s$. If $m_1 \in B_{\text{out}}(V)$, we are done. Otherwise, suppose we have a set $M_{i-1} = \{m_1, \dots, m_{i-1}\} \subset V \setminus B_{\text{out}}(V)$ such that for each $1 \leq j \leq i-1$, m_j is a non-articulation point of $V \setminus \{m_1, \dots, m_{j-1}\}$. Then $V \setminus M_{i-1}$ is connected, so as before, $V \setminus M_{i-1}$ contains at least one non-articulate module $m_i \neq s$. Set $M_i = M_{i-1} \cup \{m_i\}$.

For some minimal $t > 1$, the cell m_t found in this way must be in $B_{\text{out}}(V)$, as there are only finitely many modules in V. If m_t is a non-articulate module of V, we are again done. Otherwise, by the connectivity of $V \setminus M_t$, all of $B_{\text{out}}(V) \setminus \{m_t\}$ lies in a single connected component of $V \setminus \{m_t\}$, so m_t must have a neighboring cell not in $B_{\text{out}}(V)$. Hence, m_t must be adjacent to m_i for some $1 \leq i \leq t-1$. By Lemma 4 with $x = m_t$, all modules not in the component of m_i in $V \setminus \{m_t\}$ are in the component containing $B_{\text{out}}(V) \setminus \{m_t\}$ (note that $B_{\text{out}}(V)$ is in a single component by choice of m_t), thus removing m_t leaves exactly two components one of which is disjoint from $B_{\text{out}}(V)$. Hence, m_t is nearly non-articulate, as required. ■

Proof: By Lemma 3, V contains two non-articulate modules, and hence V has at least one non-articulate module $m_1 \neq s$. If $m_1 \in B_{\text{out}}(V)$, we are done. Otherwise, suppose we have a set $M_{i-1} = \{m_1, \dots, m_{i-1}\} \subset V \setminus B_{\text{out}}(V)$ such that for each $1 \leq j \leq i-1$, m_j is a non-articulation module of $V \setminus \{m_1, \dots, m_{j-1}\}$. Then $V \setminus M_{i-1}$ is connected, so as before, $V \setminus M_{i-1}$ contains at least one non-articulate module $m_i \neq s$. Set $M_i = M_{i-1} \cup \{m_i\}$.

For some minimal $t > 1$, the cell m_t found in this way must be in $B_{\text{out}}(V)$, as there are only finitely many modules in V. If m_t is a non-articulate module of V, we are again done. Otherwise, by the connectivity of $V \setminus M_t$, all of $B_{\text{out}}(V) \setminus \{m_t\}$ lies in a single connected component of $V \setminus \{m_t\}$, so m_t must have a neighboring cell not in $B_{\text{out}}(V)$. Hence, m_t must be adjacent to m_i for some $1 \leq i \leq t-1$. By Lemma 4 with $x = m_t$, all modules not in the component of m_i in $V \setminus \{m_t\}$ are in the component containing $B_{\text{out}}(V) \setminus \{m_t\}$, thus removing m_t leaves exactly two components one of which is disjoint from $B_{\text{out}}(V)$. Hence, m_t is nearly non-articulate, as required. ■

Lemma 7. *Given a configuration V of $n \geq 2$ modules and a module $s \in B_{\text{out}}(V)$, it is possible to reconfigure V to a configuration V', keeping $B_{\text{out}}(V)$ fixed during the reconfiguration, so that V' has a non-articulate module $x \neq s$ in $B_{\text{out}}(V') = B_{\text{out}}(V)$.*

Proof: We induct on n, the number of modules in V. The case $n = 2$ is clear. For the general case, we may find by Lemma 6 a module $x \in B_{\text{out}}(V) \setminus \{x\}$ which is either non-articulate or nearly non-articulate. In the former case, $V = V'$ and x is the chosen module.

In the latter case, let O and I be the outer and inner components of $V \setminus \{x\}$. Let $y \in I$ be the module adjacent to x; note that y is unique by Lemma 4. Also by Lemma 4, there is a module $w \notin I$ adjacent to x, which cannot be opposite from y. So, let c be the cell adjacent to the cells of y and w, which must be empty since $w \notin I$. Let f be the face of y adjacent to cell c;

it is clear that f is on the outer boundary of I (this is a direct consequence of Lemma 4). Thus, since I has fewer modules than V, the inductive hypothesis shows that we may reconfigure I to I' without moving $B_{\text{out}}(I)$ and then find a non-articulate module $m \in B_{\text{out}}(I')$ that is distinct from y. Next, as the outer boundary of I' is connected, we may find a path along the outer boundary of I' taking m to face f while avoiding the other faces of y. Move m along this path. At the first stage during these steps that m becomes adjacent to a module in O (note that this is true when m reaches f, but may occur sooner), x is no longer articulate in V, because I' and O are now connected by m. ■

Proof of Theorem 1: We show that V may be reconfigured into a straight chain. Let $s \in B_{\text{out}}(V)$ be a module with maximal x_1-coordinate, and let f be the face of s in the positive x_1 direction. Initially, denote $V_0 = V$ and $Z_0 = \{\}$. After step $i-1$ ($1 \le i \le n-1$), suppose s has not moved, and the configuration has the form $V_{i-1} \cup Z_{i-1}$, where Z_{i-1} is a straight chain of $i-1$ modules emanating from face f of s in the positive x_1 direction, V_{i-1} is connected, and $s \in B_{\text{out}}(V_{i-1})$.

By Lemma 7, we may reconfigure V_{i-1} to V'_{i-1} while keeping $B_{\text{out}}(V_{i-1})$ fixed in such a way that there is a module $x \in B_{\text{out}}(V'_{i-1})$ different from s that is non-articulate in V'_{i-1}. This implies that it is non-articulate in $V'_{i-1} \cup Z_{i-1}$, so we may simply move x along the boundary of $V'_{i-1} \cup Z_{i-1} \setminus \{x\}$ so that it extends the chain Z_{i-1}. Let Z_i be this new chain of length i, and let V_i be $V'_{i-1} \setminus x$. These clearly satisfy the above conditions, so we may repeat this process. After stage $n-1$, we are done. ■

III. Algorithm

The proof of Theorem 1 given in Section II-B gives rise to a simple algorithm to reconfigure an n-module configuration V into a straight chain. Here we present this algorithm (Algorithm 2) and prove its correctness.

We first require a recursive method that, given a configuration V and a module $s \in B_{\text{out}}(V)$ (along with a particular face of s on the outer boundary), modifies V and returns a module x according to Lemma 7. We assume that each module m has previously been assigned a field **PostOrder**(m) which sorts the modules of V in the order of finishing times of a depht-first search beginning at s. See Algorithm 1, which converts Lemma 7 to a routine **LocateAndFree**.

Most of Algorithm 1 follows Lemma 7 directly. To prove Algorithm 1 correct, we must address the comments in lines 3 and 9.

First, we must show that the module x in $B_{\text{out}}(V)$ with minimal finishing time is non-articulate or nearly

Algorithm 1 Locate a cell $x \in B_{\text{out}}(V)$ satisfying Lemma 7. Assumes V has been post-ordered.

1: **LocateAndFree**$(V, s) :=$
2: Locate all faces in V's outer boundary by depth-first search from s. We obtain $B_{\text{out}}(V)$.
3: Let $x \in B_{\text{out}}(V)$ with smallest post-order. $\{x$ is non-articulate or nearly non-articulate$\}$
4: Compute all modules in the component O of $V \setminus \{x\}$ containing s by depth-first search.
5: **if** O contains all all neighbors of x **then**
6: **return** x.
7: **else**
8: Let y be x's neighbor in the other component I.
9: Let $z =$ **LocateAndFree**(I, y). $\{$Use existing post-order labels.$\}$
10: Move z to connect O and I as in Lemma 7, locating its path by depth-first search across $V \setminus \{z\}$'s outer boundary.
11: **return** x.
12: **end if**
13: **end LocateAndFree**

non-articulate. If x is articulate in V, then a path from s to any module $t \notin O$ must pass through x, meaning t's finishing time is lower than x's. But this means t cannot be in B_{out}, by x's minimality. Thus, any connected component of $V \setminus \{x\}$ not containing s is disjoint from B_{out}, so Lemma 4 applies, proving that x is indeed nearly non-articulate.

We must also prove that the field **PostOrder** sorts the modules of I in a post-order from y. By choice of x, the original depth-first tree restricted to I must itself be a valid depth-first tree of I rooted at y, and thus the **PostOrder** field is correctly sorted, as needed.

Now we may present Algorithm 2, which rearranges V into a straight chain. The proof of correctness follows directly from the results in Section II-B.

Algorithm 2 Turn V into a straight chain $\{s\} \cup Z_{n-1}$.

Pick an extremal module $s \in V$.
Set $V_0 = V$ and $Z_0 = \{\}$.
for $1 \le i \le n-1$ **do**
Set the **PostOrder** fields with a depth-first search rooted at s.
Set $x =$ **LocateAndFree**(V_{i-1}, s).
By depth-first search across the outer boundary faces of $V_{i-1} \cup Z_{i-1} \setminus \{x\}$, move x to extend Z_{i-1}. Then set V_i and Z_i as in Theorem 1.
end for

A. Algorithm Analysis

We will analyze the number of module moves and the computation time in the above Algorithm, showing that both are asymptotically tight: $O(n^2)$ on an n-module configuration.

First, we prove by induction that **LocateAndFree**(V, s) runs in $O(|V|)$ time. As there are at most $O(|B_{\mathrm{out}}(V)|)$ faces on V's outer boundary, Lines 2 and 3 in the definition of **LocateAndFree** take $O(|B_{\mathrm{out}}(V)|) \leq O(|O|)$ time. Line 4 also runs in $O(|O|)$ time. By inductive hypothesis, Line 9 takes $O(|I|)$ time, and Line 10 takes $O(|B_{\mathrm{out}}(I)|) \leq O(|I|)$ time. The whole method thus has runtime

$$2 \cdot O(|O|) + 2 \cdot O(|I|) + O(1), = O(|V|)$$

as desired. Finally, each of the three lines in the For-loop in Line 3 in Algorithm 2 runs in $O(n)$ time, so Algorithm 2 itself has $O(n^2)$ runtime. Finally, as the module moves are made during the execution of the algorithm, there are at most $O(n^2)$ module moves as well.

To see that this is the best possible, note that it takes $O(n^2)$ module moves to transform a straight chain in one orientation to a straight chain in a different orientation, as remarked in [DP].

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