# CONFIGURATIONS OF RANK-$40r$ EXTREMAL EVEN UNIMODULAR LATTICES ($r = 1, 2, 3$)

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Abstract. We show that if $L$ is an extremal even unimodular lattice of rank $40r$ with $r = 1, 2, 3$ then $L$ is generated by its vectors of norms $4r$ and 4r+2. Our result is an extension of Ozeki's analogous result for the case $r = 1$.

## 1. Introduction

A *lattice* of *rank* $n$ is a free $\mathbb{Z}$-module of rank $n$ equipped with a positive-definite inner product $(\cdot,\cdot) : L \times L \to \mathbb{R}$. The *dual* of $L$, denoted $L^*$, is the set

$$L^* = \{y \in L \otimes \mathbb{R} : \forall x \in L, (x, y) \in \mathbb{Z}\},$$

which itself forms a lattice of the same rank as $L$. For a lattice vector $x \in L$, we call $(x, x)$ the *norm* of $x$. A lattice $L$ is *integral* if $(x, x') \in \mathbb{Z}$ for all $x, x' \in L$, i.e. if and only if $L \subseteq L^*$. An integral lattice is said to be *unimodular* if it is self-dual ($L = L^*$).

A lattice $L$ is called *even* if and only if every lattice vector has an even integer norm, i.e. $(x, x) \in 2\mathbb{Z}$ for $x \in L$. An even lattice is automatically integral by the familiar parallelogram identity, $2\,(x, x') = (x + x', x + x') - (x, x) - (x', x')$.

Lattices that are simultaneously even and unimodular are especially rare. Indeed, such a lattice's rank must be divisible by 8. Sloane proved that if $L$ is an even unimodular lattice of rank $n$ then the minimal (nonzero) norm in $L$ is bounded by

$$\min_{\substack{x \in L \\ x \neq 0}} (x, x) \leq 2\lfloor n/24 \rfloor + 2 \tag{1}$$

(see [2, p. 194, Cor. 21]). An even unimodular lattice of rank $n$ is called *extremal* if it attains the bound (1).

Ozeki [6, 8] showed that if $L$ is an extremal even unimodular lattice of rank 32 or 48 then $L$ is generated by its vectors of minimal norm. The first author [5] showed analogous results for extremal even unimodular lattices of ranks 56, 72, and 96. In a similar vein, Ozeki [7] showed that if $L$ is extremal even unimodular of rank 40, then $L$ is generated by its vectors of norms 4 and 6. Here, we extend and slightly simplify Ozeki's methods, recovering Ozeki's rank-40 result and obtaining analogous results for extremal even unimodular lattices of ranks 80 and 120.

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## 2. Modular Forms and Theta Series

We will use the notation $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ for the *upper half plane* of complex numbers. A *modular form of weight $k$ for the group $PSL_2(\mathbb{Z})$* is a holomorphic function $f : \mathcal{H} \to \mathbb{Z}$ which is holomorphic at $i\infty$ and satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL_2(\mathbb{Z})$. If a modular form $f$ vanishes at $z = i\infty$, it is called a *cusp form.*

Let $M_k$ and $M_k^0$ be the $\mathbb{C}$-vector spaces of modular forms and cusp forms of weight $k$ respectively. It is known that the *Eisenstein series*

$$E_4(z) = 1 + 240e^{2\pi iz} + 2160e^{4\pi iz} + 6720e^{6\pi iz} + \cdots \text{ and}$$
$$E_6(z) = 1 - 504e^{2\pi iz} - 16632e^{4\pi iz} - 122976e^{6\pi iz} - \cdots,$$

which are modular forms of weights 4 and 6 respectively, freely generate the spaces $M_k$ in the sense that any nonzero modular form can be written uniquely as a weighted homogeneous polynomial in $E_4$ and $E_6$. This implies that $\dim(M_k) = 0$ for $k$ odd, negative, or $k = 2$; that $\dim(M_{2k}) = 1$ and $\dim(M_{2k}^0) = 0$ for $k = 0$, $2 \le k \le 5$ and $k = 7$; and that multiplication by the weight-12 modular form $\Delta = 12^{-3}(E_4^3 - E_6^2)$ defines an isomorphism $M_{k-12} \xrightarrow{\sim} M_k^0$. More information on the theory of modular forms for $PSL_2(\mathbb{Z})$ can be found in [9].

The *theta function* $\Theta_L : \mathcal{H} \to \mathbb{Z}$ associated to a lattice $L$ is defined by

$$\Theta_L(z) = \sum_{x \in L} e^{\pi i (x,x) z};$$

it is a generating function encoding the norms of $L$'s vectors. For a homogeneous *harmonic polynomial* $P \in \mathbb{C}[x_1, \ldots, x_n]$, i.e. a homogeneous polynomial for which $\sum_{j=1}^n \frac{\partial^2 P}{\partial x_j^2} \equiv 0$, we define the *weighted theta series* $\Theta_{L,P}$ by

$$\Theta_{L,P}(z) = \sum_{x \in L} P(x) e^{\pi i (x,x) z}.$$

As shown in [9, 3], if $L$ is an even unimodular lattice of rank $n$ then $\Theta_L$ is a modular form of weight $\frac{n}{2}$, and if in addition $P$ is a homogeneous harmonic polynomial of degree $d$, then $\Theta_{L,P}$ is a modular form of weight $\frac{n}{2} + d$.

## 3. Main Result

We denote by $P_{d,x_0}(x)$ the "zonal spherical harmonic polynomial" of degree $d$, related to the *Gegenbauer polynomial* by

$$P_{d,x_0}(x) = G_d((x, x_0), ((x,x)(x_0,x_0))^{1/2}), \tag{2}$$

where $G_d(\cdot,\cdot)$ is the homogeneous polynomial of degree $d$ such that $G_d(t,1)$ is the Gegenbauer polynomial of degree $d$ evaluated at $t$ [1].

We let $L$ be an extremal even unimodular lattice of rank $40r$, and adopt the notation used by Ozeki in [7]: For an even unimodular lattice $L$, we denote by $\Lambda_{2m}(L)$ the set of vectors in $L$ having norm $2m$. We denote by $\mathcal{L}_{2m}(L)$ the sublattice of $L$ generated by $\Lambda_{2m}(L)$, and similarly denote by $\mathcal{L}_{2m_1+2m_2}(L)$ the sublattice of $L$ generated by $\Lambda_{2m_1}(L) \cup \Lambda_{2m_2}(L)$.

We define $a(2k, L) := |\Lambda_{2k}(L)|$. It is clear that the theta series $\Theta_L$ is given by $\Theta_L(z) = \sum_{k=0}^{\infty} a(2k, L)e^{2k\pi iz}$. We note that

$$4r = 2\lfloor 5r/3 \rfloor + 2 = \min\{2k > 0 : a(2k, L) \neq 0\}$$

is the minimal norm of vectors in $L$ and use the notation

$$N_j(x) = |\{y \in \Lambda_{4r}(L) : (x, y) = j\}|\,,$$
$$M_j(x) = |\{y \in \Lambda_{4r+2}(L) : (x, y) = j\}|\,.$$

Using the involution $y \longleftrightarrow -y$ of $\Lambda_m(L)$, we see that we have $N_j(x) = N_{-j}(x)$ and $M_j(x) = M_{-j}(x)$ for any $j \in \mathbb{R}$ and $x \in L \otimes \mathbb{R}$.

We will show the following configuration result, which directly extends Ozeki's [7] result for extremal even unimodular lattices of rank 40:

**Theorem 3.1.** *For $r = 1, 2, 3$ and $L$ extremal even unimodular of rank $40r$, we have $L = \mathcal{L}_{4r+(4r+2)}(L)$.*

*Proof.* We partition $L$ into its equivalence classes modulo $\mathcal{L}_{4r+(4r+2)}(L)$. We need only show that any class $[x] \in L/\mathcal{L}_{4r+(4r+2)}(L)$ is represented by a vector $x_0 \in [x]$ with $(x_0, x_0) \leq 4r + 2$.

Now, we suppose there exists some equivalence class $[x_0] \in L/\mathcal{L}_{4r+(4r+2)}(L)$ where $x_0 \neq 0$ is a representative of minimal norm with $(x_0, x_0) = 2t$ for some $t \geq 2r + 2$. We have the inequality

$$|\,(x_0, x)\,| \leq 2r \text{ for all } x \in \Lambda_{4r}(L), \tag{3}$$

as $x_0$ is not minimal in $L$ whenever $(x_0, \pm x) > 2r$ since the vector $x \mp x_0$ has norm

$$(x \mp x_0, x \mp x_0) = (x, x) \mp 2\,(x, x_0) + (x_0, x_0) < (x_0, x_0)\,.$$

Similarly, we have

$$|\,(x_0, x)\,| \leq 2r + 1 \text{ for all } x \in \Lambda_{4r+2}(L). \tag{4}$$

From (3) and (4), we have the equations

$$\sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2k} = \sum_{j=1}^{2r} 2 \cdot j^{2k} \cdot N_j(x_0), \tag{5}$$

$$\sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2k} = \sum_{j=1}^{2r+1} 2 \cdot j^{2k} \cdot M_j(x_0), \tag{6}$$

for all $k > 0$.

We extract from the theta series $\Theta_L$ of $L$ the coefficients $a(4r, L)$ and $a(4r+2, L)$. We observe immediately from (5) and (6) that

$$\sum_{x \in \Lambda_{4r}(L)} (x, x_0)^0 = a(4r, L), \tag{7}$$

$$\sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^0 = a(4r + 2, L). \tag{8}$$

Since $L$ is even unimodular of rank $40r$, we have $\Theta_{L,P_{d,x_0}} \in M^0_{20r+d}$ for any $d > 0$. By comparing power-series coefficients, we then observe

$$\Theta_{L,P_{d,x_0}} \equiv 0 \text{ for } d \in \{2, \ldots, 4r-2, 4r+2\}, \tag{9}$$

$$\Theta_{L,P_{4r,x_0}} \equiv c_1 \Delta^{2r} \text{ for a constant } c_1, \tag{10}$$

$$\Theta_{L,P_{4r+4,x_0}} \equiv c_2 E_4 \Delta^{2r} \text{ for a constant } c_2. \tag{11}$$

From (9), we obtain the equations

$$\sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2d} = a(4r, L) \frac{1 \cdot 3 \cdots (2d-1)}{40r \cdot (40r+2) \cdots (40r+2d-2)} (8r)^d t^d \quad \text{and} \tag{12}$$

$$\sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2d} = a(4r+2, L) \frac{1 \cdot 3 \cdots (2d-1)}{40r \cdot (40r+2) \cdots (40r+2d-2)} (8r+4)^d t^d, \tag{13}$$

for $d \in \{2, \ldots, 4r-2, 4r+2\}$. We obtain from (10)

$$\sum_{x \in \Lambda_{4r+2}(L)} P_{4r,x_0}(x) = c_{4r} \sum_{x \in \Lambda_{4r}(L)} P_{4r,x_0}(x), \tag{14}$$

where $\Delta^{4r} = e^{(4r)\pi i z} + c_{4r} e^{(4r+1)\pi i z} + O(e^{(4r+2)\pi i z})$. Similarly, (11) gives

$$\sum_{x \in \Lambda_{4r+2}(L)} P_{4r+4,x_0}(x) = c_{4r+4} \sum_{x \in \Lambda_{4r}(L)} P_{4r+4,x_0}(x), \tag{15}$$

where $E_4 \Delta^{4r} = e^{(4r)\pi i z} + c_{4r+4} e^{(4r+1)\pi i z} + O(e^{(4r+2)\pi i z})$.

Combining the equations (7), (8), (12), (13), (14), and (15) with (5) and (6), we obtain a system of $4r+4$ homogeneous linear equations in the $4r+3$ unknowns

$$N_0(x_0), \ldots, N_{2r}(x_0), M_0(x_0), \ldots, M_{2r+1}(x_0).$$

At this stage, we diverge from our natural generalization of Ozeki's original methods and obtain the (extended) determinants of these inhomogeneous linear systems; these determinants must vanish because the system is overdetermined.

For $r = 1, 2, 3$, these determinants are respectively

$$2^{55} 3^7 5^8 7^4 11^4 13^1 19^6 23^3 \cdot (t-2) \cdot t \cdot (6t-13) \cdot \left(10t^2 - 55t + 77\right), \tag{16}$$

$$2^{132} 3^{27} 5^{16} 7^{10} 11^6 13^{10} 23^4 41^8 43^6 47^3 \cdot (t-4) \cdot t \cdot Q_2(t), \tag{17}$$

$$2^{244} 3^{48} 5^{26} 7^{13} 11^7 13^7 17^6 23^4 31^{11} 37^1 59^{14} 61^{11} 67^5 71^3 73^1 \cdot (t-6) \cdot t \cdot Q_3(t), \tag{18}$$

where $Q_2(t)$ is the irreducible quintic

$$10768t^5 - 242280t^4 + 2202310t^3 - 10101795t^2 + 23361877t - 21771246$$

and $Q_3(t)$ is the irreducible septic

$$\begin{gathered} 1998988267405690 9935t^7 - 892881426107875310430t^6 \\ +1725803960122265415 1533t^5 - 18705331032112190430 6075t^4 \\ +12273982499082291814 23784t^3 - 48740109459092638103 20032t^2 \\ +10840974078436271024 624064t - 10414527769923133690 990080. \end{gathered}$$

In each case, there are no integer solutions $t \geq 2r + 2$. However, we had assumed the existence of an equivalence class

$$[x_0] \in L/\mathcal{L}_{4r+(4r+2)}(L)$$

with minimal-norm representative $x_0 \neq 0$ having $(x_0, x_0) = 2t$ for integral $t \geq 2r+2$; since no such $t$ exists, all equivalence classes must be generated by vectors having norms $4r$ and $4r + 2$. □

## 4. Concluding Remarks

A quick inspection will show that our results are the only possible immediate extensions of Ozeki's methods. In the cases $r \geq 4$, it is not possible to extract sufficiently many linear conditions by these exact techniques, as the dimensions of the relevant spaces of cusp forms grows too large.

However, using different analysis, Elkies [4] has shown a stronger result than our Theorem 3.1 in the $r = 3$ case: If $L$ is an extremal unimodular lattice of rank 120 then $L = \mathcal{L}_{12}(L)$. This result for rank-120 lattices is analogous to Ozeki's [6, 8] results in dimensions 32 and 48, and to the first author's [5] results in dimensions 56, 72, and 96.

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