A SUMMARY OF *THE SECOND LEFSCHETZ THEOREM ON HYPERPLANE SECTIONS*, BY ALDO ANDREOTTI AND THEODORE FRANKEL

ZACHARY ABEL

In this paper we give a guided tour of the proof of the second Lefschetz hyperplane section theorem as presented in [2]. The present writeup is based primarily on my notes used while presenting the Andreotti-Frankel argument to the Harvard University, Mathematics 99r Morse Theory seminar in Spring 2009. As such, this work uses much of the same notation as, and we follow the same general structure as, [2].

1. Introduction

Suppose we have a smooth, n-dimensional algebraic subvariety $X \subset \mathbb{P}^N(\mathbb{C}) := \mathbb{P}$. The first Lefschetz hyperplane section theorem states that for a generic hyperplane section X_0, the map $H_i(X_0, \mathbb{Z}) \to H_i(X, \mathbb{Z})$ is an isomorphism for $i < n-1$ and is surjective if $i = n-1$. Many nice proofs exist, such as [1] by the same authors. The second Lefschetz hyperplane section theorem fully describes the kernel of $H_{n-1}(X_0, \mathbb{Z}) \to H_{n-1}(X, \mathbb{Z})$. Here we show a proof of the second Lefschetz theorem using such geometric tools as Morse theory and blow-up varieties.

Our primary aid for studying hyperplane sections of X will be a *pencil* of hyperplanes of $\mathbb{P}$, namely the one-dimensional family of hyperplanes that all share some $n-2$-dimensional lineal subspace of $\mathbb{P}$. In particular, suppose two hyperplanes $P_0, P_\infty \subset \mathbb{P}$ have been chosen and that homogeneous coordinates $[Z_0, \ldots, Z_n]$ on $\mathbb{P}$ are defined so that $P_0 = \{Z_0 = 0\}$ and $P_\infty = \{Z_1 = 0\}$.[1] We may then define a holomorphic function $\varphi : \mathbb{P} \setminus (P_0 \cap P_\infty) \to \mathbb{P}^1$ so that $\mathbb{P}^1$ paramatrizes the pencil in the following sense: For $r \in \mathbb{P}^1$, $\varphi^{-1}(r)$ gives the hyperplane $\{Z_0/Z_1 = r\}$, on in other words, for $x \in \mathbb{P} \setminus (P_0 \cap P_\infty)$, $\varphi(x)$ denotes "which" hyperplane x belongs to. The restriction $\varphi|_X$ thus paramatrizes the hyperplane sections belonging to the pencil in the same sense.

The map φ is not regular, however, as it is not defined on the common intersection of the pencil, namely the $n-2$ linear subspace $P_0 \cap P_\infty$. Because of this, we instead focus attention to a space $\tilde{X} \twoheadrightarrow X$, called the *blowup of* X, with an everywhere-defined map $\tilde{X} \to \mathbb{P}^1$ extending φ.

2. Blowing up X

Use the same notation as in the previous section, with the following assumption regarding the chosen hyperplanes P_0 and P_∞:

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[1]Throughout the proof, we will assume that the chosen P_0 and P_∞ satisfy three Assumptions to be stated later. It is not difficult to show that planes P_0 and P_∞ may always be found that satisfy these three hypotheses; however for the sake of clarity and brevity we have omitted this argument.

Assumption 1. *We assume that neither P_0, P_∞, nor the $N-2$-dimensional subspace $P_0 \cap P_\infty$, is tangent to X at any point.*

Define $X_0 := X \cap P_0$, $X_\infty := X \cap P_\infty$, and $Y = X_0 \cap X_\infty$.

We define the *blowup of $\mathbb{P}$ along $P_0 \cap P_\infty$* as the submanifold $\tilde{P} \subset \mathbb{P} \times \mathbb{P}^1$ defined as

$$\tilde{P} := \left\{ \left([Z_0, \dots, Z_N], [t_0, t_1] \right) \,\middle|\, Z_0 t_1 = Z_1 t_0 \right\}.$$

Let $\pi^N : \mathbb{P}^N \times \mathbb{P}^1 \to \mathbb{P}^N$ be the projection onto the first coordinate, and let $\tilde{\pi} : \tilde{X} \to X$ be the restriction $\tilde{\pi} = \pi^N\big|_{\tilde{\mathbb{P}}}$. Similarly, let $\pi^1 : \mathbb{P}^N \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection onto the second factor. The projection $\tilde{\pi}$ is a biholomorphism away from $P_0 \cap P_\infty$; indeed, if $(Z_0, Z_1) \neq (0,0)$ then $\tilde{\pi}^{-1}([Z_0, \dots, Z_N])$ is the single point $([Z_0, \dots, Z_N], [Z_0, Z_1])$. If $(Z_0, Z_1) = (0,0)$, then

$$\tilde{\pi}^{-1}([Z_0, \dots, Z_N]) = \left\{ ([Z_0, \dots, Z_N], [t_0, t_1]) \,\middle|\, [t_0, t_1] \in \mathbb{P}^1 \right\} \simeq \mathbb{P}^1.$$

Thus, $\tilde{\pi}^{-1}(P_0 \cap P_\infty) \simeq (P_0 \cap P_\infty) \times \mathbb{P}^1$. The map $\pi^1\big|_{\tilde{\mathbb{P}}}$ is the promised holomorphic extension of φ: indeed, if $(Z_0, Z_1) \neq (0,0)$, then $\pi^1\left([Z_0, \dots, Z_N], [t_0, t_1]\right) = [t_0, t_1] = [Z_0, Z_1]$.

We may now consider the blowup of X: because of Assumption 1, $\tilde{\pi}$ is *transverse regular* on X, and therefore $\tilde{X} := \tilde{\pi}^{-1}(X) \subset \tilde{\mathbb{P}}$ is a smooth, n-dimensional submanifold. Let $\pi := \tilde{\pi}|_{\tilde{X}}$. As before, π is biholomorphic on $X \setminus Y$, and $\pi^{-1}(Y) =: \tilde{Y}$ is biholomorphic with $Y \times \mathbb{P}^1$ via π. Finally, let $f := \pi^1\big|_{\tilde{X}}$, and note that for each $[t_0, t_1] \in \mathbb{P}^1$, $f^{-1}([t_0, t_1]) \subset \tilde{X}$ is biholomorphic with the hyperplane section $X \cap \varphi^{-1}([t_0, t_1])$ via π. For convenience, let $\mathcal{X}_0 := f^{-1}([0,1])$ and $\mathcal{X}_\infty := f^{-1}([1,0])$.

3. Critical points of f

In preparation for an invocation of Morse theory, we study the critical points of $f : \tilde{X} \to \mathbb{P}^1$. First, let $\tilde{y} = ([0, 0, Z_2, \dots, Z_N], [t_0, t_1]) \in \tilde{Y}$ be any point. Let $\hat{f}$ denote the function f restricted to the line $[0, 0, Z_2, \dots, Z_N] \times \mathbb{P}^1$. Note that $\hat{f}$ is an isomorphism, so $\mathrm{d}\hat{f}|_{\tilde{y}}$ is subjective, so $\mathrm{d}f|_{\tilde{y}}$ is surjective. Thus, $\tilde{y}$ cannot be a critical point of f, *i.e.* f has no critical points in $\tilde{Y}$.

Now suppose $\tilde{x} \in \tilde{X} \setminus \tilde{Y}$ is a critical point of f. Then because π near x is a biholomorphism, $x := \pi(\tilde{x})$ is a critical point of φ, and it can be shown that the latter occurs if and only if the hyperplane $\varphi^{-1}(\varphi(x))$ is tangent to X at x. But $\varphi^{-1}(\varphi(x))$ is one of the planes in our pencil through $P_0 \cap P_\infty$, so we conclude that the critical points of f correspond precisely (via π) to those points where a plane in the pencil through $P_0 \cap P_\infty$ is tangent to X. We make a second assumption:

Assumption 2. *The planes in the pencil through $P_0 \cap P_\infty$ are tangent to X in finitely many (say μ) points.*

Because P_0 and P_∞ are assumed not to be tangent to X, this implies that $\mathcal{X}_0$ and $\mathcal{X}_\infty$ (which surject onto X_0 and X_∞ respectively through π) contain no critical points of f.

Let us compute f locally at one of these critical points $\tilde{x}$. If $f(x) = [t_0, t_1]$, then let $\alpha := t_0/t_1$, which by the previous line is neither 0 nor ∞. Take affine coordinates $(\zeta_0, \dots, \zeta_{N-1}) :=$

$(Z_0/Z_1, Z_2/Z_1, \ldots, Z_N/Z_1)$ for X around $x := \pi(\tilde{x})$, so that X is tangent to the plane $\zeta_0 = \alpha$ at x. By a linear change of coordinates we may assume that $\zeta_1, \ldots, \zeta_n$ form a system of local coordinates for X at x and that $\zeta_1(x) = \cdots = \zeta_n(x) = 0$. As X is tangent to the plane $\zeta_0 = \alpha$ at x, we may write $\varphi(\zeta_1, \ldots, \zeta_n) = \alpha + \sum_{i,j=1}^n a_{ij}\zeta_i\zeta_j + \text{h.o.t.}$, where h.o.t. stands for "higher order terms." We may apply another linear change of coordinates to $\zeta_1, \ldots, \zeta_n$ (by diagonalizing and scaling the matrix (a_{ij})) in order to write

$$\varphi(\zeta_1, \ldots, \zeta_n) = \alpha + \alpha \cdot \sum_{i=1}^r \zeta_i^2 + \text{h.o.t.} \tag{1}$$

for some $0 \le r \le n$. As this process preserves the singularity or invertibility of the matrix (a_{ij}), we find that $\det(a_{ij}) = 0$—*i.e.*, x is a *degenerate* critical point of φ—if and only if $r < n$. Our third assumption states that we always have $r = n$:

Assumption 3. *We assume that each critical point of f, i.e., the inverse image through π of each point of tangency between X and a plane in the pencil containing P_0 and P_∞, is a nondegenerate critical point of f.*

Note that Assumption 3 formally implies Assumption 2 because $\tilde{X}$ is compact (as it is a projective subvariety of $\mathbb{P}^N$) and nondegenerate critical points are isolated.

4. Morse structure of $\tilde{X}$

Define $\nu : \mathbb{P}^1 \setminus \{\infty\} \to \mathbb{R}$ by $\nu([t_0, t_1]) = |t_0/t_1|^2$. Now that we have analyzed the critical points of $f : \tilde{X} \to \mathbb{P}^1$, we can use the real-valued function $F := \nu \circ f : \tilde{X} \setminus \mathcal{X}_\infty \to \mathbb{R}$ to determine the topological structure of $\tilde{X}$.

Because ν has a single critical point at $[0, 1]$, the critical points of F are either points $\tilde{x}$ where $F(\tilde{x}) = 0$—namely, the points in $\mathcal{X}_0$—or one of the μ critical points of f. Around one of the critical points of f, we may take local complex coordinates $\zeta_1, \ldots, \zeta_n$ so that equation 1 holds near $\tilde{x}$, and then the real coordinates $\zeta_j = \sqrt{2}(u_j + iv_j)$ reveal that

$$F(u_1, v_1, \ldots, u_N, v_N) = |\alpha|^2 \cdot \left(\sum_{i=1}^n u_i^2 - v_i^2 \right) + \text{h.o.t.}$$

locally. As there are n positive terms and n negative terms in this representation, the *index* of f at $\tilde{x}$ is exactly n.

Finally, let $D_\varepsilon = \left\{ t \in \mathbb{P}^1 \,\middle|\, \nu(t) \le \varepsilon \right\}$. For ε small enough, I claim that $f^{-1}(D_\varepsilon)$ is topologically $\mathcal{X}_0 \times D_\varepsilon$ and has a deformation retract onto $\mathcal{X}_0$. Indeed, this follows because $[0, 1]$ is a regular value of f. We may now invoke Morse theory: as each critical point of F outside of $\mathcal{X}_0 \cup \mathcal{X}_\infty$ is nondegenerate of degree n, it follows from [3, Remark 3.3] that $\tilde{X} - \mathcal{X}_\infty$ has the same homotopy type as

$$\tilde{X} \setminus \mathcal{X}_\infty \sim \mathcal{X}_0 \cup e_n^1 \cup \cdots \cup e_n^\mu,$$

where each e_n^i is an n-cell attached along its boundary to the $(n-1)$-skeleton of some triangulation of $\mathfrak{X}_0$.

5. Comparing X with $\tilde{X}$

As our ultimate goal is to understand the topology of X and its hyperplane sections, we now justify the attention given to $\tilde{X}$ by studying the induced map $\pi_* : H_k(\tilde{X}) \to H_k(X)$.

Because $\tilde{Y} \sim Y \times \mathbb{P}^1$, we may define a map $\beta : H_k(Y) \to H_{k+2}(\tilde{Y})$ given by $z_k \mapsto z_k \times P^1$. Letting $i : \tilde{Y} \hookrightarrow \tilde{X}$ be the inclusion, we may define a map $\sigma : H_k(Y) \to H_{k+2}(\tilde{X})$ as $\sigma = (-1)^k i_* \circ \beta$. The specific relation that we will prove between X and $\tilde{X}$ is that the following is a split short exact sequence for all k:

$$0 \longrightarrow H_k(Y) \xrightarrow{\sigma} H_{k+2}(\tilde{X}) \xrightarrow{\pi_*} H_{k+2}(X) \longrightarrow 0 \tag{2}$$

First, because π has degree 1, it is known that the map $h : H_j(X) \to H_j(\tilde{X})$ given by $h = \lambda_{\tilde{X}} \circ \pi^* \circ \lambda_X^{-1}$ (where λ_Q denotes the Poincaré duality map on Q) is a right inverse to $\pi_* : H_j(\tilde{X}) \to H_j(X)$ for each j, so in particular, π_* is surjective.

Let $J(\tilde{X}, X) = \tilde{X} \times [0,1] \cup X / \{(\tilde{x}, 1) \sim \pi(\tilde{x})\}$ be the mapping cylinder of $\pi : \tilde{X} \to X$. Define the subspace $A := (J(\tilde{X}, X) \setminus J(\tilde{Y}, Y)) \cup \tilde{Y}$. The map $r : A \to \tilde{X}$ given by $(\tilde{x}, t) \mapsto \tilde{x}$ is well defined because π is one-to-one away from $\tilde{Y}$ and thus each point $x \in X \setminus Y \subset A$ has a unique preimage $\pi^{-1}(x) \in \tilde{X}$. Further, r is a deformation retract, and hence induces an isomorphism on homology groups. Likewise, the deformation retract $u : J(\tilde{X}, X) \to X$ defined by $(\tilde{x}, t) \mapsto \pi(\tilde{x})$ induces an isomorphism on homology. In the following commutative diagram, the top line is the long exact sequence for relative homology, and the isomorphisms r_* and u_* are as above:

$$\begin{array}{ccccccccccc}
\cdots \to & H_{k+3}(A) & \twoheadrightarrow & H_{k+3}(J(\tilde{X},X)) & \to & H_{k+3}(J(\tilde{X},X),A) & \xrightarrow{\partial} & H_{k+2}(A) & \twoheadrightarrow & H_{k+2}(J(\tilde{X},X)) & \to \cdots \\
& \simeq \downarrow r_* & & \simeq \downarrow u_* & & & & \simeq \downarrow r_* & & \simeq \downarrow u_* & \\
& H_{k+3}(\tilde{X}) & \xrightarrow{\pi_*} & H_{k+3}(X) & & & & H_{k+2}(\tilde{X}) & \xrightarrow{\pi_*} & H_{k+2}(X) &
\end{array}$$

Because both occurrences of π_* are surjective in the above diagram, it follows that the top row in the diagram below is short exact. By excision on the subset $J(\tilde{X}, X) \setminus J(\tilde{Y}, Y)$, we obtain isomorphism α_1. The rest of the diagram will be explained presently.

$$\begin{array}{ccccccccc}
0 \to & H_{k+3}(J(\tilde{X},X),A) & \xrightarrow{r_* \circ \partial} & H_{k+2}(\tilde{X}) & \xrightarrow{\pi_*} & H_{k+2}(X) & \to 0 \\
& \simeq \downarrow \alpha_1 & \nearrow r_* \circ \partial & \uparrow \sigma & & & \\
& H_{k+3}(J(\tilde{Y},Y),\tilde{Y}) & & & & & \\
& \simeq \uparrow T & & & & & \\
& H_k(Y) & & & & &
\end{array} \tag{3}$$

For the last part of the diagram, we use the Thom Isomorphism:

Theorem 1 (Thom Isomorphism)**.** *Let $p : E \to B$ is an n-dimensional real vector bundle, and let $D(E)$ and $S(E)$ denote the corresponding disk and sphere bundles on B. Then we have an isomorphism $\Phi : H_{i+n}(B) \to H_i(D(E), S(E))$ given by $\Phi(b) = p_*(b) \frown c$. Here, $c \in H^n(D(E), S(E))$ is a Thom class, meaning its restriction to each fiber is a generator of $H^n(D^n, S^{n-1})$.*

In our specific case, note that $\tilde{Y} \simeq Y \times \mathbb{P}^1$ is a real sphere bundle over Y formed by a product with a 2-sphere $\mathbb{P}^1 \simeq S^2$. On each fiber, the mapping cylinder $J(\tilde{Y}, Y)$ simply forms the cone between the given point and S^2, thus creating the product disc bundle $J(\tilde{Y}, Y) \simeq Y \times D^3$. So in this simple case, the Thom Isomorphism above may be shown to produce the isomorphism $T : H_k(Y) \xrightarrow{\simeq} H_{k+3}(J(\tilde{Y}, Y), \tilde{Y})$ given by $T(z_k \times e_3)$, where e_3 is a fixed fiber of $Y \times D^3$. Finally, computing $\partial(z_k \times e_3) = (-1)^k z_k \times \partial e_3$ shows that $r_* \circ \partial \circ T = \sigma$, which explains the final element in equation 3. This proves the short exact sequence claimed in equation 2.

6. Finally! A proof of the second Lefschetz theorem

Recall from Section 4 that $\tilde{X} \setminus \mathcal{X}_\infty \simeq \mathcal{X}_0 \cup e_n^1 \cup \cdots \cup e_n^\mu$. In this section, all homology rings are over $\mathbb{Z}$. Let the subgroup of $H_{n-1}(\mathcal{X}_0)$ generated by the boundaries of $e_n^1, \ldots, e_n^\mu$ be called the *vanishing cycles on* $\mathcal{X}_0$, and note (by definition of cellular homology, for example) that the kernel of the map $H_{n-1}(\mathcal{X}_0) \to H_{n-1}(\tilde{X} \setminus \mathcal{X}_\infty)$ consists of exactly these vanishing cycles. Similarly, let the images $(\pi|_{\mathcal{X}_o})_*(\partial e_n^1), \ldots, (\pi|_{\mathcal{X}_o})_*(\partial e_n^\mu)$ generate a subgroup of X_0 called the *vanishing cycle group on* X_0; the second Lefschetz Theorem states the following:

Theorem 2 (Second Lefschetz Theorem)**.** *The map $H_{n-1}(X_0) \to H_{n-1}(X)$ is surjective, and the kernel is exactly the vanishing cycle group on X_0.*

Proof. This result follows by combining all of our previous work into a single, beautiful diagram chase. As such, we must first construct the diagram.

Define $\mathcal{Y} := \tilde{Y} \cap \mathcal{X}_\infty$, which can be thought of as $Y \times \{\infty\} \subset Y \times \mathbb{P}^1 \simeq \tilde{Y}$ and can thus be identified with Y. The diagram is as follows:

$$
\begin{array}{ccccccccc}
 & & H_{n-3}(\mathcal{X}_\infty) & \xrightarrow[\simeq]{\tilde{T}} & H_{n-1}(\tilde{X}, \tilde{X} \setminus \mathcal{X}_\infty) & & & & \\
 & & \ell_* \uparrow \simeq & & k_* \uparrow & & & & \\
0 & \longrightarrow & H_{n-3}(\mathcal{Y}) & \xrightarrow{\sigma} & H_{n-1}(\tilde{X}) & \xrightarrow{\pi_*} & H_{n-1}(X) & \longrightarrow & 0 \\
 & & & & j_* \uparrow & & i_* \uparrow & & \\
 & & & & H_{n-1}(\tilde{X} \setminus \mathcal{X}_\infty) & \xleftarrow[\tau_*]{} & H_{n-1}(X_0) & & \\
 & & & & \partial \uparrow & & & & \\
 & & H_{n-2}(\mathcal{X}_\infty) & \xrightarrow[\simeq]{\tilde{T}} & H_n(\tilde{X}, \tilde{X} \setminus \mathcal{X}_\infty) & & & & \\
 & & \ell_* \uparrow & & k_* \uparrow & & & & \\
 & & H_{n-2}(\mathcal{Y}) & \xrightarrow{\sigma} & H_n(\tilde{X}). & & & &
\end{array}
$$

Most of these maps are straightforward: the middle column is the long exact sequence for $\tilde{X}$ relative to $\tilde{X} \setminus X_\infty$; the second row is the short exact sequence (2) after the identification $\mathcal{Y} \simeq Y$; τ is the map $\tau : X_0 \xrightarrow{\simeq} \mathcal{X}_0 \hookrightarrow \tilde{X} \setminus \mathcal{X}_\infty$; and $\ell : \mathcal{Y} \hookrightarrow \mathcal{X}_\infty$ is just the inclusion. The map $\tilde{T}$ is slightly more involved, and invokes the Thom isomorphism (Theorem 1) again.

As we did for $\mathcal{X}_0$ in Section 4, we may find a small tubular neighborhood $N_\infty = \mathcal{X}_\infty \times D$ of $\mathcal{X}_\infty$ in $\tilde{X}$, where D is a closed 2-disc. Let $\dot{N}_\infty$ be the corresponding circle bundle. It follows from excision that $H_k(\tilde{X}, \tilde{X} - \mathcal{X}_\infty) \simeq H_k(N_\infty, \dot{N}_\infty)$. By Theorem 1, we have an isomorphism $T : H_k(\mathcal{X}_\infty) \to H_{k+2}(N_\infty, \dot{N}_\infty)$, so we define $\tilde{T} : H_k(\mathcal{X}_\infty) \xrightarrow{T\cdot(-1)^k} H_{k+2}(N_\infty, \dot{N}_\infty) \xrightarrow{\simeq} H_k(\tilde{X}, \tilde{X} \setminus \mathcal{X}_\infty)$.

Now that the diagram is properly defined, we must verify commutativity. The square $i_* = \pi_* \circ j_* \circ \tau_*$ is clear, as $\pi \circ j \circ \tau$ simply includes X_0 into $\mathcal{X}_0$ and then maps this back onto $X_0 \subset X$, whereas i just includes $X_0 \hookrightarrow X$. The two other squares present more of a challenge; we may compute that a cycle $z_k \in H_k(\mathcal{Y})$ is sent by $k_* \circ \sigma$ to $(-1)^k z_k \times S \in H_{k+2}(\tilde{X}, \tilde{X} \setminus \mathcal{X}_\infty)$, whereas—by tracing the Thom isomorphism carefully—z_k is sent to $z_k \times e_2'$ where $e_2' = S \cap N_\infty$ is a closed 2-disk. As in [2], these two cycles can be seen to be equivalent modulo $\tilde{X} \setminus \mathcal{X}_\infty$.[2]

One more property of the diagram will be useful: $\ell_* : H_{n-3}(\mathcal{Y}) \to H_{n-3}(\mathcal{X}_\infty)$ is an isomorphism, and $\ell_* : H_{n-2}(\mathcal{Y}) \to H_{n-2}(\mathcal{X}_\infty)$ is surjective. These follow from the first Lefschetz hyperplane theorem: $\mathcal{X}_\infty \simeq X_\infty$ is a smooth $n-1$ manifold and $\mathcal{Y} \simeq Y \simeq X_\infty \cap X_0$ is a hyperplane section of X_∞.

We may finally proceed with the promised diagram chase. We must show that $\ker(\tau_*) = \ker(i_*)$. One direction is clear: if $\tau_*(z) = 0$ for some $z \in H_{n-1}(X_0)$, then $\iota_*(z) = \tau_* \circ j_* \circ \pi_*(z) = \tau_* \circ j_*(0) = 0$, so $\ker \tau_* \subset \ker i_*$. Conversely, suppose $i_*(z) = 0$ for some $z \in H_{n-1}(X_0)$. Because $\pi_* \circ j_* \circ \tau_*(z) = i_*(z) = 0$, by exactness of the second column there must exist some $y \in H_{n-3}(\mathcal{Y})$ with $\sigma(y) = j_* \circ \tau_*(z)$. We then have $0 = k_* \circ j_*(\tau_*(z)) = \tilde{T} \circ \ell_*(y)$, and because $\tilde{T} \circ \ell_*$ is an isomorphism, $y = 0$. It follows that $j_*(\tau_*(z)) = j_*(0) = 0$. Now we make our way to the bottom half of the diagram: by exactness of the middle column, because $j_*(\tau_*(z)) = 0$ there is some $x \in H_n(\tilde{X}, \tilde{X} \setminus \mathcal{X}_\infty)$ mapping to $\tau_*(z)$. Because $\tilde{T} \circ \ell_*$ is surjective, choose any $w \in H_{n-2}(\mathcal{Y})$ mapping onto x. Then $\tau_*(z) = \partial \circ k_*(\sigma(w)) = 0$ by exactness of the middle column at $H_n(\tilde{X}, \tilde{X} \setminus \mathcal{X}_\infty)$. This completes the chase and the proof of the second Lefschetz hyperplane theorem. □

References

[1] Aldo Andreotti and Theodore Frankel. The lefschetz theorem on hyperplane sections. *The Annals of Mathematics*, 69(3):713–717, 1959.

[2] Aldo Andreotti and Theodore Frankel. The second Lefschetz theorem on hyperplane sections. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 1–20. Univ. Tokyo Press, Tokyo, 1969.

[3] J.W. Milnor. *Morse Theory*. Princeton Univ. Press, Princeton, N.J., 1963.

[2] Detail has been omitted here; see [2] for a more careful discussion.