TANGLES FOR KNOTS AND LINKS

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1. Introduction

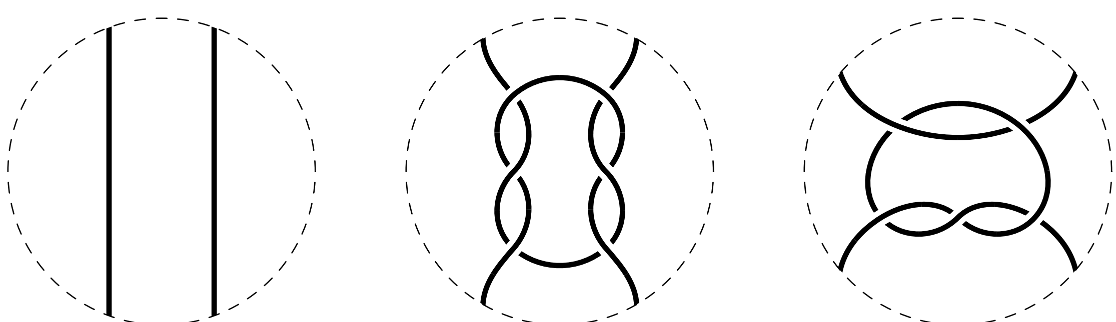
It is often useful to discuss only small "pieces" of a link or a link diagram while disregarding everything else. For example, the Reidemeister moves describe manipulations surrounding at most 3 crossings, and the skein relations for the Jones and Conway polynomials discuss modifications on one crossing at a time. Tangles may be thought of as small pieces or a local pictures of knots or links, and they provide a useful language to describe the above manipulations. But the power of tangles in knot and link theory extends far beyond simple diagrammatic convenience, and this article provides a short survey of some of these applications.

2. Tangles As Used in Knot Theory

Formally, we define a tangle as follows (in this article, everything is in the PL category):

Definition 1. A ***tangle*** is a pair (A, t) where $A \cong B^3$ is a closed 3-ball and t is a proper 1-submanifold with $\partial t \neq \emptyset$. Note that t may be disconnected and may contain closed loops in the interior of A. We consider two tangles (A, t) and (B, u) ***equivalent*** if they are isotopic as pairs of embedded manifolds. An ***n-string tangle*** consists of exactly n arcs and $2n$ boundary points (and no closed loops), and the ***trivial n-string tangle*** is the tangle $(B^2, \{p_1, \ldots, p_n\}) \times [0, 1]$ consisting of n parallel, unintertwined segments.

See Figure 1 for examples of tangles. Note that with an appropriate isotopy any tangle (A, t) may be transformed into an equivalent tangle so that A is the unit ball in $\mathbb{R}^3$, $\partial t = A \cap t$ lies on the great circle in the xy-plane, and the projection $(x, y, z) \mapsto (x, y)$ is a regular projection for t; thus, tangle diagrams as shown in Figure 1 are justified for all tangles.



(a) The trivial 2-string tangle, T_1. (b) The clasp, known here as T_2. (c) Tangle T_3.

Figure 1.

We may now formalize the intuition that a tangle is a "piece" of a knot or link.

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Definition 2. The ***tangle sum*** of two links (A,t) and (B,u) along a homeomorphism $\phi : \partial(A,t) \to \partial(B,u)$ is the link obtained by gluing (A,t) and (B,u) along ϕ; specifically, it is the link $t \cup_\phi u$ embedded in $A \cup_\phi B \cong S^3$.

Note that the resulting link depends greatly on the homeomorphism ϕ; for example, Figure 2 shows that two trivial 2-string tangles may be added in many ways.

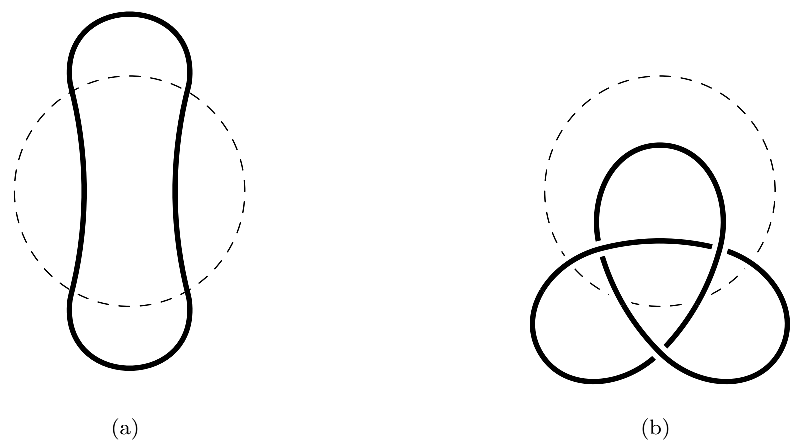


FIGURE 2. Two different ways to form a tangle sum with two trivial 2-string tangles.

Tangle decompositions are extremely useful for studying properties of the decomposed knots or tangles. For example, Conway [1] used tangle decompositions to provide a convenient link notation that allowed him to "list (by hand) all knots of 11 crossings or less and all links of 10 crossings or less" as well as to compute useful invariants from these decompositions. Below we see two more examples of tangle decompositions facilitating knot and link theory.

2.1. **Bridge Number.** Here we give an example of tangle decompositions allowing the *definition* of a useful knot/link invariant:

Definition 3. The *bridge number* $b(L)$ of a link L is the smallest integer n such that L may be expressed as the tangle sum of two trivial n-string tangles.

To see that this number is well defined, note that any link diagram with n arcs readily produces a decomposition into two trivial n-string tangles, which means not only does $b(L)$ exists but that it is bounded above by the crossing number of L.

It is not difficult to see that any link L with $b(L) = 1$ may be drawn without crossings and is hence the unknot. Slightly less straightforward is the following theorem:

Theorem 1. Any knot K with $b(K) = 2$ is prime.

(Such knots with bridge number 2 are called 2-*bridge knots*.) While Theorem 1 has an elementary proof (Exercise!), it also follows from the following difficult theorem of Schubert:

Theorem 2. If a knot K is factored as $K = K_1 \# K_2$, then $b(K) = b(K_1) + b(K_2) - 1$.

2.2. **Proving Nonsplittability of Links.** We saw in the previous subsection that tangle decompositions are useful for *defining* knot/link invariants. Here we will see how they can be useful for *proving* link invariants, specifically, the nonsplittability of a link:

Definition 4. A link $L \subset S^3$ is *nonsplit* if it is impossible to find a sphere $S^2 \subset S^3$ disjoint from L which separates L.

We will show that the tangle sum of two nonsplit tangles is always a nonsplit link.

Definition 5. A tangle (A, t) is *nonsplit* if any disk $D \cong B^2$ properly embedded in A and disjoint from t does not separate t.

Theorem 3. If (A, t) and (B, u) are nonsplit tangles, then any link $L = (A, t) \cup_\phi (B, u)$ is a nonsplit link.

Proof. The proof is a simple cut-and-paste argument. Let L sit inside $S^3 \simeq A \cup_\phi B$ and let $F \cong S^2 \subset S^3$ be the image of ∂A (equivalently, ∂B in S^3. Suppose $G \cong S^2 \subset S^3$ is a 2-sphere disjoint from L; we must show that G does not separate components of L.

First suppose $F \cap G = \emptyset$, and say WLOG that $G \subset A$. If G split t in A, then by connecting G and ∂A with a small cylinder we would obtain a splitting disk embedded in A, which does not exist because A is nonsplit. Thus, $G \subset A$ cannot split $t \subset A$ and thus cannot split L, as needed.

Now suppose $F \cap G$ is nonempty. We may assume WLOG that G intersects F transversely, and thus their intersection is a finite collection of simple closed loops on G. Choose such a closed loop $\gamma \in F \cap G$ that is *innermost* on G, bounding the empty disk $g \cong B^2 \subset G$. Suppose WLOG that $g \subset A$. Because A is nonsplit, disk g divides A into two 3-balls, one of which is empty. We may then isotope G through this empty sub-ball of A in order to remove the arc γ from the intersection $F \cap G$ (and possibly other intersection arcs as well), thus strictly reducing the number of components of $F \cap G$. Repeat this until $F \cap G = \emptyset$, and finish as above. □

3. Studying Tangles

The previous section illustrates that understanding tangles is crucial for understanding knots and links, so in this section we focus on studying tangles in their own right. As with knots and links, our primary objectives are to distinguish and classify tangles. We begin by proving the existence of nontrivial tangles:

Theorem 4. Tangles T_2 and T_3 from Figure 1 are nonsplit and therefore distinct from the trivial 2-string tangle, T_1.

Proof. If T_2 were split, then because its strands are unknotted, it would be the trivial 2-string tangle. This means that adding any other 2-string tangle would result in a 2-bridge knot. But it is clearly possible to add a trivial 2-string tangle to T_2 and obtain $3_1 \# 3_1^*$, which is not prime and hence by Theorem 1 does not have bridge number 2.

If T_3 were split, then in any knot K written as a tangle sum involving T_3, the knotted arc of T_3 would provide a factor of a trefoil, proving that K decomposes as $K = 3_1 \# K'$ for some K'. But it is clearly possible to add a trivial 2-string tangle to T_3 to obtain the unknot, which is a contradiction. □

While these arguments are not difficult, they are *ad-hoc* and not directly generalizable to many other tangles. We therefore turn to more widely applicable invariants and theorems surrounding tangle classification.

3.1. **Double.** For a tangle T, let T^* be the mirror image of T, and let $\phi : \partial T \to \partial T^*$ be the induced homeomorphisms on the boundaries. Then the link double$(T) := T \cup_\phi T^*$, as illustrated in Figure 3, is called the *double* of tangle T.

Theorem 5. The double of a tangle T is a well-defined tangle invariant.

Proof. Indeed, if T and U are equivalent tangles via isotopy ψ, then the map $\psi \cup_\phi \psi^*$ is easily seen to be an ambient isotopy from $T \cup_\phi T^*$ to $U \cup_\phi U^*$. □

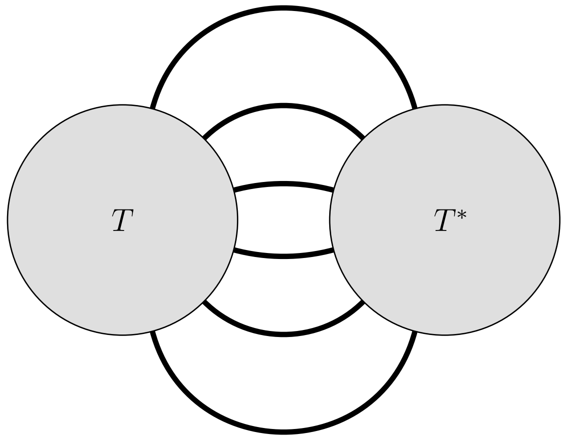


FIGURE 3. The double of a tangle T is obtained by gluing a diagram D to its mirror image.

The tangle double may be used to provide a quick alternate proof the nonsplittability of the clasp:

Proof that T_2 is nonsplit. If T_2 were split, then it would be the trivial 2-string tangle. But the doubles of T_2 and T_1 may be distinguished by their Jones polynomials:

$$V(\text{double}(T_1)) = [-1, \quad \text{and} \quad V(\text{double}(T_2)) = [-1, -2, 3, -2, 2, -1,$$

where $[a_1, a_2, \ldots$ is shorthand for $a_1(q^{1/2} + q^{-1/2}) + a_2(q^{3/2} + q^{-3/2}) + \cdots$. □

Note that the double is not a perfect tangle invariant; indeed, Figure 4 shows two different tangles (one has 4 boundary components while the other has 8) with the same double. Nevertheless, the double is a very powerful invariant.

3.2. **Krebes Number.** The Krebes invariant, though only applicable to tangles with four boundary points, has the advantage of being numerical and very simple to compute. For a tangle T with four boundary points, pick a diagram D for T with boundary points in the north-east, north-west, south-east, and south-west corners of the bounding circle, and form the *numerator* D^n and *denominator* D^d as shown in Figure 5. Then the *Krebes invariant* of T is defined as $\gcd(\det(D^n), \det(D^d))$. For a proof of well-definedness, see [3].

The Krebes invariant may be used to provide yet another simple proof of the nonsplittability of the clasp:

Proof that T_2 is nonsplit. The numerator of the diagram for T_2 in Figure 1 is the unlink which has determinant 0. The denominator is $3_1 \# 3_1^*$, and we may compute $\det(T_2^d)$ as follows (using Δ_K for the Alexander polynomial of knot K):

$$\det(3_1 \# 3_1^*) = \Delta_{3_1 \# 3_1^*}(-1) = \left(\Delta_{3_1} \cdot \Delta_{3_1^*}\right)\big|_{-1} = (1 - t + t^2)^2\big|_{t=-1} = 9.$$

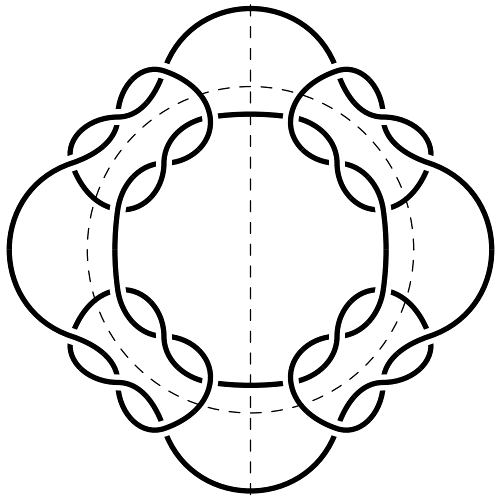
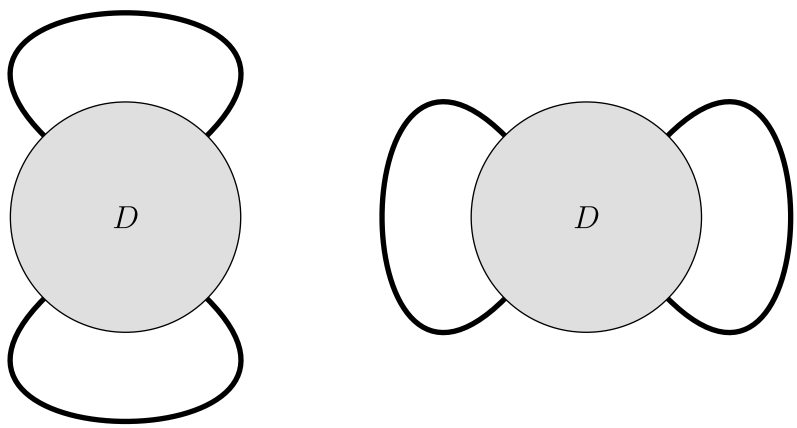


FIGURE 4. This link is the double of two different tangles: we may take the boundary between D and D^* to be either the dotted circle (thus forming the double of a tangle with 8 boundary points) or the dotted line (the double of a tangle with 4 boundary points).



(a) The numerator, D^n. (b) The denominator, D^d.

FIGURE 5.

So $\mathrm{kr}(T_2) = 9$. On the other hand, $\mathrm{kr}(T_1)$ is easily computed to be 1, proving that $T_1 \neq T_2$, as needed. □

It should be noted that the tangle double and the Krebes invariant alone are sufficient to fully classify all 4-boundary-point tangles with up to 7 crossings, as in [2]. There are 39 such tangles.

3.3. **Proving Nonsplittability of Tangles.** Finally, we illustrate a powerful way to demonstrate the nonsplittability of many tangles. The following theorem is similar to Theorem 3:

Theorem 6. Let (C, v) be a tangle and D a disk properly embedded in C such that D divides (C, v) into two tangles (A, t) and (B, u). We assume the following:

(1) The number of points in $(\partial A - D) \cap v$, $(\partial B - D) \cap v$, and $D \cap v$ are all at least 1;
(2) (A, t) and (B, u) are nonsplit.

Then (C, v) is nonsplit.

This may be proved with a cut-and-paste argument similar to, yet more involved than, that of Theorem 3. For full details, see [4].

4. Fun Application and Concluding Thoughts

Our concluding example is meant to further emphasize the importance of tangle theory in the context of knots and links: we use our knowledge of tangles to easily prove the existence of Brunnian links of any size.

Theorem 7. Define an n-component *Brunnian link* as an n-component link such that all proper sublinks are totally split, *i.e.* all components may be separated from each other by disjoint spheres. Then n-component Brunnian links exist for any $n \geq 2$.

Proof. Let U_1 be the clasp, and for each $k \geq 2$ define U_k as the tangle formed by joining the clasp with U_{k-1} as shown in Figure 6. Because the clasp is nonsplit, it follows by induction and Theorem 6 that U_k is nonsplit for each k.

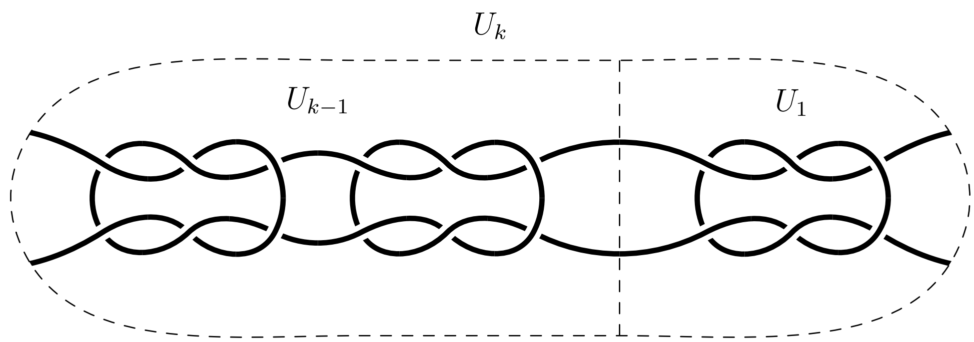


Figure 6. Constructing U_k inductively from U_{k-1} and U_1 (shown above for $k = 3$). By induction, U_k in nonsplit for each $k \geq 1$.

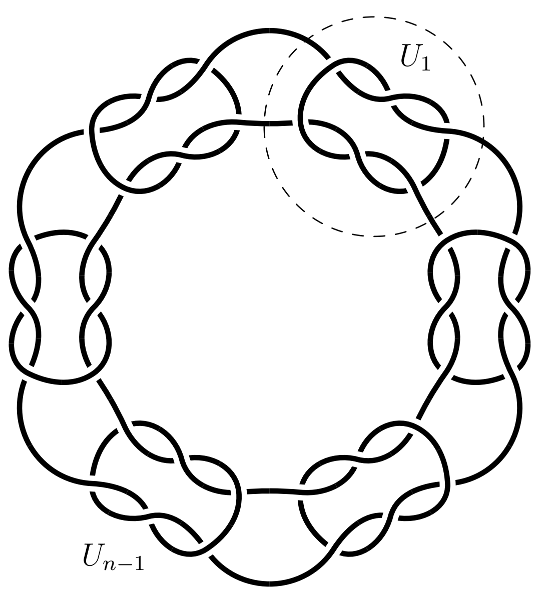


Figure 7. Constructing L_n from U_{n-1} and U_1 (shown above for $n = 6$). This is a Brunnian link on n components.

For each $n \geq 2$, define $L_n := U_1 \cup_\phi U_{n-1}$ where the gluing map ϕ is illustrated in Figure 7. By Theorem 3, this link L_n is nonsplit. However, if any one component of L_n is removed, it is not difficult to see that the remaining link is totally split, so L_n is indeed a Brunnian link, as desired. □

This brief survey does not do justice to the ubiquity of tangle theory in the study of knots and links. A great source for further reading is Nakanishi's thesis, "Links and Tangles" [4].

References

[1] Conway, J. H. "An Enumeration of Knots and Links, and Some of Their Algebraic Properties." In J. Leech (editor), *Computational Problems in Abstract Algebra.* Oxford, England. Pergamon Press, 329–358, 1970.

[2] Kanenobu, T., Saito, H, and Satoh, S. "Tangles with up to Seven Crossings." *Interdisciplinary information sciences*, **9.1**: 127–140, 2003.

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[4] Nakanishi, Y. "Links and Tangles." `http://www.lib.kobe-u.ac.jp/repository/thesis/d1/D1000396.pdf`.