THETA SERIES AS MODULAR FORMS

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1. Introduction

In this paper we introduce the notion of the **theta series** Θ_L of a lattice L, a useful and powerful tool in Lattice theory, especially in the case when the underlying lattice is assumed to be even and unimodular. Recall that a lattice L is **even** if for all vectors $x \in L$, the norm $\langle x, x\rangle = \|x\|^2$ is an even integer, and the lattice L of rank n is **unimodular** if its fundamental domain has volume $\operatorname{vol}(\mathbb{R}^n/L) = 1$, or equivalently, if $L = L^*$.

In section 2 we define the theta series of a lattice L and provide an example for the lattice $L = A_2$. We introduce the **modular group** and **modular forms** in section 3, and then prove that when L is an even unimodular lattice, Θ_L assumes the structure of a modular form. Results relating to **Eisenstein series** and the classification of modular forms are detailed in 4, and finally, we illustrate the power of theta series by proving the uniqueness of E_8 as a unimodular lattice of rank 8 in section 5.

2. What is a Theta Series?

2.1. **Definition.** For a given lattice L, we define a generating function that counts the number of vectors in L having a given norm. Specifically, if A_r denotes the number of vectors in L with norm r, we define

$$\Theta_L(q) = \sum_{x\in L} q^{\frac{1}{2}\langle x,x\rangle} = \sum_{\ell} A_{2\ell} q^{\ell}. \tag{1}$$

The reason for the factor of $\frac{1}{2}$ in the exponent will shortly become evident.

A lattice's theta series encodes the distribution of vector norms in the lattice L. Note however that it does not encode *all* information about the lattice: for example, the nonisomorphic lattices $E_8 \oplus E_8$ and D_{16}^+ have identical theta series, so L cannot always be reconstructed uniquely from Θ_L.

2.2. **An Example: A_2.** As an example, consider $\Theta_{A_2}(q)$, the theta series of the 2-dimensional hexagonal lattice. From Figure 1, we see that there is a single vector of norm 0, six of norm 1, none of norm 2 (remember that vectors of length 2 have norm $2^2 = 4$), 6 of norm 3, etc. Therefore, the theta series begins

$$\Theta_{A_2}(q) = 1 + 6q^{\frac{1}{2}} + 0q^1 + 6q^{\frac{3}{2}} + 6q^2 + \cdots.$$

In general, the coefficient of $q^{\frac{n}{2}}$ will be the number of ordered pairs of integers (a, b) such that $\left|a + b\cdot e^{\frac{\pi i}{3}}\right|^2 = a^2 + ab + b^2 = n$. However, this only gives a characterization of $\Theta_{A_2}(q)$ using information from the lattice A_2 itself. In order to make use of the theta series, we would prefer to learn about A_2 from its theta series, not the other way.

Thus, we seek other methods for computing theta series. In the special case when the lattice L is assumed even and unimodular, such a computation is particularly easy when one considers the theta series not as a formal *algebraic* object — a generating function — but instead as an analytic function. We write $q = e^{2\pi i z}$

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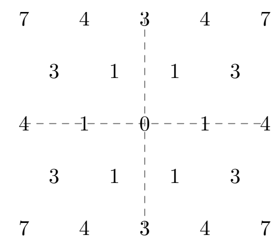

FIGURE 1. Vector norms in the Hexagonal lattice.

for z in the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ of complex numbers, and consider Θ_L as a function of z:

$$\Theta_L(z) = \sum_{x \in L} e^{2\pi i z \cdot \frac{1}{2}\langle x, x\rangle} = \sum_{\ell} A_\ell e^{\pi i z \ell}.$$

The usefulness comes from the following property:

Theorem 1. *If L is an even unimodular lattice of rank n, then $\Theta_L(z)$ is a modular form of weight $\frac{n}{2}$.*

This Theorem is explained and proven in the following section.

3. What are Modular Forms?

3.1. **Modular Group.** Intuitively, modular forms are functions that behave well under a certain action from the **modular group**. So we must first discuss the modular group G.

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the Poincaré upper half plane as above. The group $SL_2(\mathbb{Z})$ consisting of 2×2 matrices over $\mathbb{Z}$ with determinant 1 acts on $\mathcal{H}$ by means of the following map:

$$SL_2(\mathbb{Z}) \times \mathcal{H} \to \mathcal{H}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \mapsto \frac{az+b}{cz+d}.$$

This indeed maps into $\mathcal{H}$, because for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$, which is positive if $\mathrm{Im}(z)$ itself is. As both $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $-I_2$ act trivially on $\mathcal{H}$, there is an induced action of $SL_2(\mathbb{Z})/\{\pm I_2\} = G$ on $\mathcal{H}$. This group G is the modular group. (As a notational convention, a matrix $M \in SL_2(\mathbb{Z})$ is also used to denote its coset $\{M, -M\} \in G$.)

We single out two elements of G:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; z \mapsto -\frac{1}{z}, \qquad \text{and} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; z \mapsto z+1.$$

Geometrically, S acts by the inversion in the unit circle composed with reflection in the imaginary axis, and T shifts one unit to the right. These elements can be seen to satisfy $S^2 = 1$ and $(TS)^3 = 1$.

The region $D = \{z \in \mathcal{H} \mid -\frac{1}{2} \le \mathrm{Im}(z) \le \frac{1}{2} \text{ and } |z| \ge 1\}$, shaded in Figure 2, is known as the **fundamental domain** for the Modular group, for the following reason (stated without proof):

Theorem 2. [4, p.78] *For every $z \in \mathcal{H}$, there exists $g \in G$ such that $gz \in D$. Furthermore, if two points z and z' of D are congruent modulo G, then $\mathrm{Re}(z) = \pm\frac{1}{2}$ and $z' = z \pm 1$, or $|z| = 1$ and $z' = -1/z$, or $z = z'$.*

In figure 2, the repeated actions of S and T on D are illustrated. Graphically, it seems that the actions of S and T on D cover the entire half-plane, i.e. for every point $z \in \mathcal{H}$, there is some element g in the ideal generated by S and T so that $gz \in D$. In fact, much more can be said:

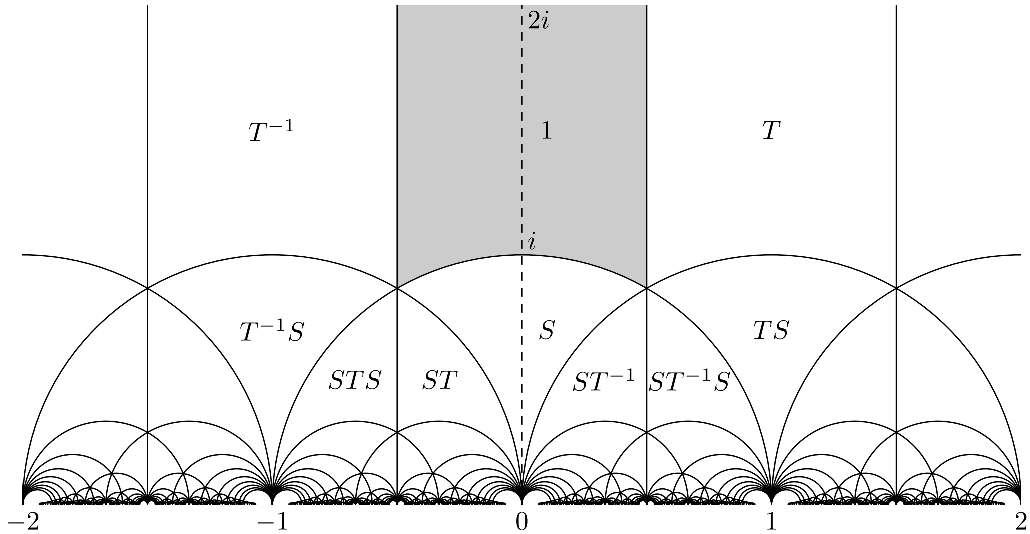

FIGURE 2. The fundamental domain for the action of G on the half plane.

Theorem 3. *The elements S and T generate the modular group.*

Proof. Let G' be the ideal generated by S and T; we wish to show that $G' = G$. We begin by proving the fact stated above, i.e. for every $z \in \mathcal{H}$ there is some element $g \in G'$ with $gz \in D$. Suppose that for some $z = x + iy$ this was not the case. Let a be an integer so that $-\frac{1}{2} \leq a + x \leq \frac{1}{2}$, i.e. $-\frac{1}{2} \leq \mathrm{Re}(T^a(z)) \leq \frac{1}{2}$. If $y \geq 1/2$, then $T^a(z)$ is in one of the regions labelled 1, S, ST, or ST^{-1} in figure 2, so if g is this label, then $g^{-1} \cdot T^a \in G'$ is the desired element of G' that sends z into D. Otherwise, consider $ST^a(z)$. Its imaginary part is $\frac{y}{(a+x)^2+y^2} \geq \frac{y}{\frac{1}{4}+y^2} \geq 2y$, where the final inequality relies on $y \leq \frac{1}{2}$. Thus, we can repeat this process until the imaginary part is at least $\frac{1}{2}$, and then finish as before.

Now we can prove the Theorem. Given an element $m \in G$, consider the value $m(2i) \in \mathcal{H}$. By the above property, there is some element $g \in G'$ so that $g(m(2i)) \in D$. This means that $2i$ and $gm(z)$ — both elements of D — are equivalent modulo G, and since $2i$ is not on the boundary of D, it follows that $gm(2i) = 2i$, from which it follows easily that $gm = 1$. Thus, $m = g^{-1}$ is in G', as desired. □

In fact, it can be shown that S and T give the following presentation for the modular group:

$$G = \langle S, T \mid S^2, (TS)^3 \rangle.$$

3.2. **Modular Forms.** Intuitively, a modular form is a function f on $\mathcal{H}$ that behaves well under the action of the modular group. Formally, a modular form of weight k (for a non-negative even integer k) it is a holomorphic[1] — i.e. everywhere complex differentiable — function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following two properties:

(a). $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \cdot f(z)$, and

(b). f has a power series expansion $\sum_{n=0}^{\infty} a_n q^n$ in the variable $q = e^{2\pi i z}$, i.e., f is holomorphic at infinity.

Since G is generated by elements S and T, property (a). above is equivalent to the following two properties:

(a.i). $f(Tz) = f(z+1) = f(z)$, and

[1]Notions from Complex Analysis have been suppressed in this paper. See [4] for reference.

(a.ii). $f(Sz) = f\left(-\frac{1}{z}\right) = z^k \cdot f(z)$.

3.3. **Even Unimodular Lattices.** In this section we'll prove the Theorem promised on page 2:

Theorem 1. *If L is an even unimodular lattice of rank n, then $\Theta_L(z)$ is a modular form of weight $\frac{n}{2}$.*

Note that

$$\Theta_L(z+1) = \sum_{x\in L} e^{2\pi i z \frac{1}{2}\langle x,x\rangle} \cdot e^{2\pi i \frac{1}{2}\langle x,x\rangle} = \sum_{x\in L} e^{2\pi i z \frac{1}{2}\langle x,x\rangle} = \Theta_L(z),$$

so property (a.i) above holds easily. Likewise, from the definition in (1) and the fact that L is even, it is clear that $\Theta_L(z)$ is a power series in q, i.e. property (b) holds. It remains to show property (a.ii). While doing this, we'll also prove (and need!) a related fact:

Theorem 4. *If L is an even unimodular lattice of rank n, then $n \equiv 0 \bmod 8$.*

Finally, to prove these Theorems, we'll need the following result of Jacobi relating the theta series of a lattice L to that of its dual.

Theorem 5 (Jacobi's Identity)**.** *For any lattice L, the following identity holds:*

$$\Theta_L\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{\frac{n}{2}} \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)} \cdot \Theta_{L^*}(z).$$

This is a corollary of the celebrated Poisson summation formula; for details, see [2, p.47].

Proof of Theorem 4. Suppose L is even unimodular of rank n, and suppose n is not divisible by 8. The lattices $L \oplus L$ or $L \oplus L \oplus L \oplus L$ are of rank $2n$ and $4n$ respectively, and they are also even unimodular. Indeed, if A is a generator matrix for L, then

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

are generator matrices of $L \oplus L$ and $L \oplus L \oplus L \oplus L$ respectively, and clearly both of these matrices have even elements along the diagonal and determinant 1 if A itself does. Thus, by replacing L by $L \oplus L$ or $L \oplus L \oplus L \oplus L$ if necessary, we may assume that the rank n is congruent to 4 mod 8.

Since $L^* = L$ and $\mathrm{vol}(\mathbb{R}^n/L) = 1$, Theorem 5 applied to L becomes $\Theta_L(Sz) = -z^{\frac{n}{2}}\Theta_L(z)$, and since we also know that Θ_L is invariant under action by T, this implies

$$\Theta_L((TS)z) = -z^{\frac{n}{2}}\Theta_L(z).$$

Iterating this three times gives

$$\Theta_L(z) = \Theta_L((TS)^3) = -((TS)^2 z)^{\frac{n}{2}}\Theta_L((TS)^2 z) = \cdots = -(-1)^{\frac{n}{2}}\Theta_L(z) = -\Theta_L(z).$$

But this implies that $\Theta_L(z)$ is identically zero, i.e. L is empty! Thus, n must indeed be divisble by 8. □

Now we can complete the proof of Theorem 1.

Proof of Theorem 1. Recall that we have shown properties (a.i) and (b), so it simply remains to demonstrate property (a.ii). But with the knowledge that n is divisble by 8, this follows directly from Jacobi's identity: $\Theta_L\left(-\frac{1}{z}\right) = z^{\frac{n}{2}}\Theta_L(z)$. □

4. Space of Modular Forms

Now that we know that an even unimodular lattice gives rise to a modular form, we will explore what is known about modular forms themselves.

4.1. **Eisenstein Series.**

The simplest examples of modular forms are the **Eisenstein series**. For an even integer $k \geq 4$, consider the following sum:

$$G_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}.$$

Theorem 6. *The Eisenstein series $G_k(z)$ is a modular form of weight k for any even integer $k \geq 4$.*

Proof. We will verify properties (a.i) and (a.ii), and refer the interested reader to [2, p.49] for verification of the holomorphicity properties. We can calculate

$$G_k(Tz) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(z+1)+n)^k} = \sum_{\substack{(m',n')\in\mathbb{Z}^2 \\ (m',n')\neq(0,0)}} \frac{1}{(m'z+n')^k} = G_k(z),$$

by making the substitution $m' = m$ and $n' = m + n$, which is (a.i). Likewise,

$$G_k(Sz) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(-\frac{m}{z}+n)^k} = z^n \cdot \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(zn-m)^k} = z^n \cdot G_k(z),$$

where the final equality follows by shifting indices as above. This is (a.ii). □

It is useful to scale these series to have constant term 1 when written as a power series in q. To calculate the required scaling factor, write $G_k(z) = \sum_{m\geq 0} a_m q^m$, and substitute $z = it$ for a real variable t:

$$G_k(it) = a_0 + \sum_{m\geq 1} a_m \cdot e^{-\pi t m}.$$

As t limits to zero this tends to the desired a_0, so we find

$$a_0 = \lim_{t\to\infty} G_k(it) = \lim_{t\to\infty} \sum_{(m,n)\neq(0,0)} \frac{1}{(mit+n)^k} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^k} = 2\zeta(k).$$

Thus we define the **normalized Eisenstein Series** E_k as $\frac{1}{2\zeta(k)} G_k$. Much can be shown about these series. For example, it can be shown that

Theorem 7. [2, p.52] *For an even integer $k \geq 4$,*

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m,$$

where B_k is the kth Bernoulli number and $\sigma_\ell(m) = \sum_{d|m} d^\ell$ is the sum of the ℓth powers of the positive divisors of m. In particular, we have

$$E_4(z) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \quad \textit{and} \quad E_6(z) = 1 - 504 \sum_{m=1}^{\infty} \sigma_5(m) q^m.$$

4.2. **Classification.**

If f_1 and f_2 are modular forms of weight k and $c \in \mathbb{C}$ is a scalar, then it is clear that $c \cdot f_1$ and $f_1 + f_2$ are also modular forms of weight k. This turns the space M_k of modular forms of weight k

into a vector space over $\mathbb{C}$. Likewise, if g is a modular form of weight ℓ, then $f_1 \cdot g$ is a form of weight $k+\ell$, and this map turns $M = \bigoplus_{k=0}^{\infty} M_k$ into a graded algebra.

The Eisenstein series provided a series of simple examples of modular forms, but in fact, in one sense, they encompass all possible modular forms! Indeed, we have the following complete classification of modular forms:

Theorem 8. [2, p.60] *The algebra M of modular forms is isomorphic to the polynomial algebra $\mathbb{C}[E_4, E_6]$ of complex polynomials in the Eisenstein series E_4 and E_6, i.e., $M = \mathbb{C}[E_4, E_6]$.*

In other words, any modular form of weight k is uniquely expressible as a weighted homogeneous polynomial in E_4 and E_6. In particular, this implies that each M_k is a finite dimensional space. This is made more explicit in the next Theorem.

Let M_k^0 denote the space of **cusp forms** of weight k, i.e. the space of weight-k modular forms with constant term 0. Also define $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, which is a weight-12 cusp form. Then we have the following explicit classification:

Theorem 9. [2, p.59]

(i). *We have $M_k = 0$ for k odd, for $k < 0$, and for $k = 2$.*

(ii). *We have $M_0 = \mathbb{C}$, $M_0^0 = 0$, and for $k = 4, 6, 8, 10$, $M_k^0 = 0$, $M_k = \mathbb{C} \cdot E_k$.*

(iii). *Multiplication by Δ defines an isomorphism of M_{k-12} onto M_k^0.*

5. Application: Uniqueness of E_8

As an example, we can use theta series analysis to prove the uniqueness of the E_8 lattice in the following sense:

Theorem 10. *If L is an even unimodular lattice of rank 8, then $L \cong E_8$.*

Proof. [3]. Suppose L is even unimodular of rank 8. Then Θ_L must be a modular form of weight 4 by Theorem 1. The classification in Theorem 9 says that M_4 is a one dimensional vector space spanned by the Eisenstein series E_4, so $\Theta_L = c \cdot E_4$ for some $c \in \mathbb{C}$. But since the constant term of Θ_L is 1 (the number of vectors in L with norm 0), we must have $c = 1$, so that

$$\Theta_L = E_4 = 1 + 240q + 2160q^2 + 6720q^3 + \cdots .$$

Thus, L has 240 vectors of norm 2, 2160 vectors of norm 4, etc.

The sublattice $2L$ is a lattice of volume $2^8 = 256$, so the group $L/2L$ is of size 256. For $x \in L$, let L_x be the coset represented by x in $L/2L$. We will enumerate those cosets represented by vectors of norm at most 4.

The trivial coset L_0, namely $2L$ itself, contains no vectors of norm 2 or 4, as the smallest nonzero vectors in $2L$ have norm at least 8.

Next consider the coset represented by a root v. If w is another root then $v - w \in 2L$ implies that $v - w$ has norm 0 or 8, which can happen if and only if $v = w$ or $v = -w$. Conversely, both v and $-v$ are indeed in L_v. Furthermore, if $a \in L$ is a vector of norm 4, then I claim $a \notin L_v$. Indeed, if $a - v \in 2L$ then we must also have $(a - v) + 2v = a + v \in 2L$, and since $a \pm v$ are both nonzero, we would have $16 \le |v - a|^2 + |v + a|^2 = 2|v|^2 + 2|a|^2 = 12$, contradiction. Thus, as there are 240 vectors of norm 2, there are $\frac{240}{2} = 120$ cosets represented by roots, and furthermore these cosets do not include any norm 4 vectors.

Finally consider the coset represented by a norm 4 element a. If $b \in L$ is another norm 4 vector, then $|b-a|^2 \leq (|a|+|b|)^2 = 16$, so if $b \in L_a$ we must have $|b-a|^2 = 0, 8,$ or 16. In the first case $a = b$, in the second case a and b are orthogonal, and in the third case $a = -b$. So the norm 4 elements in L_a are mutually orthogonal or parallel, and so there are at most 16 norm 4 vectors in this coset, since these 16 would form an orthogonal frame for $\mathbb{R}^8$. As there are 2160 vectors in L of norm 4, there are at least $\frac{2160}{16} = 135$ such cosets.

Since $1+120+135 = 256$, the requisite number of cosets, the characterization above must include them all. Further, we find that each coset L_a with $|a|^2 = 4$ must indeed have 16 elements of norm 4 in an orthogonal frame. Pick any one of these frames $a_1, \ldots, a_8, -a_1, \ldots, -a_8$. The sublattice spanned by $a_1, \ldots, a_8$, namely $(2\mathbb{Z})^8$, is certainly a sublattice of L. In fact, since $a_i \equiv a_j \bmod 2L$, we have $a_i - a_j = 2v$ for some $v \in L$, i.e. $\frac{1}{2}(a_i + a_j) = a_j + v \in L$. This implies $D_8 \subset L$, or (since L is unimodular), $D_8 \subset L \subset D_8^*$. But D_8 is an index 4 sublattice of D_8^*, so it can be easily calculated that there are only two unimodular lattices between D_8 and D_8^*, namely $D_8 \cup \left(D_8 + \frac{1}{2}(a_1 + \cdots + a_8)\right)$ and $D_8 \cup \left(D_8 + \frac{1}{2}(-a_1 + a_2 + \cdots + a_8)\right)$. Both of these are isomorphic to $D_8^+ \cong E_8$. □

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