TWO SOLUTIONS TO A TILING PROBLEM

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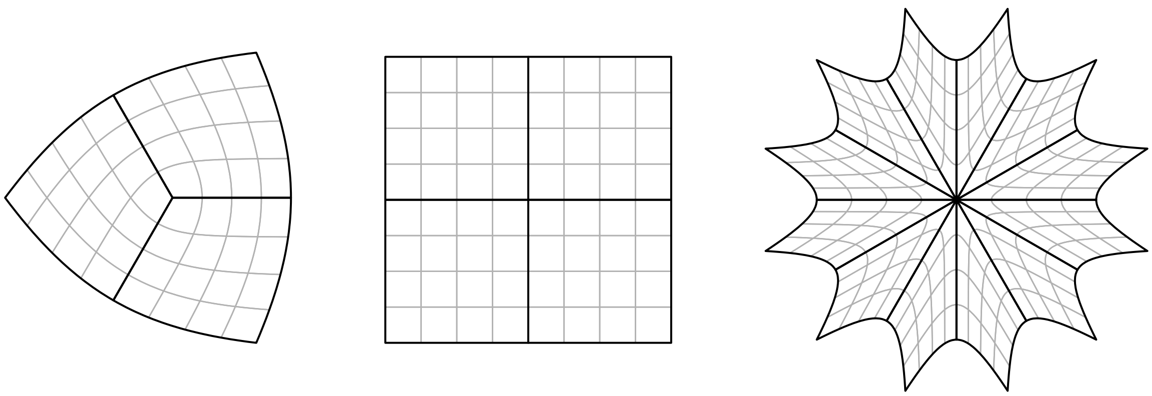
Problem. Consider three pairwise adjacent faces of an $n \times n \times n$ cube. For what values of n is it possible to tile the three faces with 3×1 "band-aids"? A band-aid may wrap around from one face to another, but cannot bend.

Answer. The faces may be tiled if and only if $3 \mid n$. One direction is easy: if $3 \mid n$ then each of the faces may be individually tiled with band-aids. It remains to show the converse: if a tiling exists, then n is divisible by 3. We offer two solutions, the second of which may be derived from the first.

Solution 1. This solution is motivated by the attempt to convert a tiling on the given surface to a tiling on a flat surface, because tilings on rectangles or squares are well understood. Indeed, assuming the given surface can be tiled, the key idea is to "wrap" four copies of this tiling into a triple tiling of a $2n \times 2n$ square, and then to show that such a triple tiling can only exist when $3 \mid n$. Details follow.



(a) *The surface S_3, drawn with smooth transitions between the three faces.*

(b) *The surface S_4 is a $2n \times 2n$ square.*

(c) *The surface S_{12}.*

Figure 1: *The surfaces S_3, S_4, and S_{12} (shown with n = 4), drawn with smooth transitions between faces.*

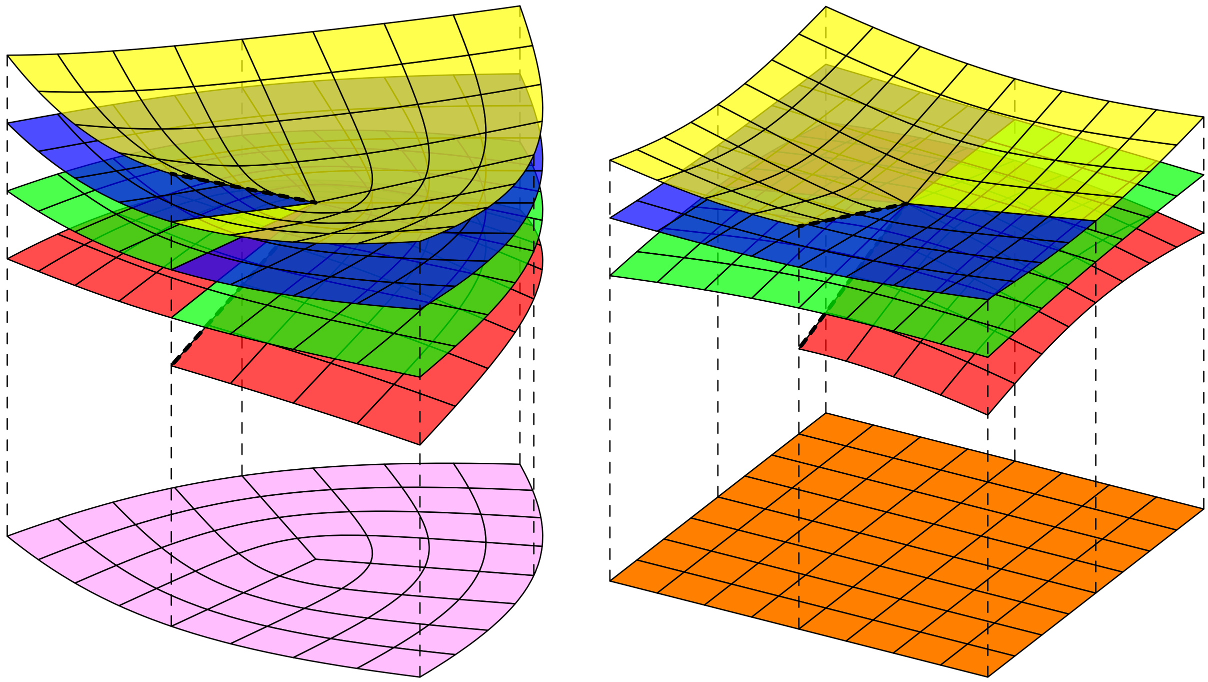
Let S_k be the surface obtained by gluing k squares—or *quadrants*—together at a common point, so that the given surface is S_3 (Figure 1(a)), S_4 is a $2n \times 2n$ square (Figure 1(b)), and S_{12} is as shown in Figure 1(c).

Suppose we know a tiling T_3 of S_3 with n^2 band-aids. There is a map $p_3 : S_{12} \to S_3$ as shown in Figure 2(a) which sends the i^{th} quadrant of S_{12} to the $(i \bmod 3)^{\text{th}}$ quadrant of S_3. If we lift each tile in T_3 to its four preimages on S_{12}, we obtain a tiling T_{12} of S_{12} by $4n^2$ band-aids. Similarly, the map $p_4 : S_{12} \to S_4$ as in Figure 2(b), which sends the i^{th} quadrant of S_{12} to the $(i \bmod 4)^{\text{th}}$ quadrant of S_4, maps the tiling T_{12} into a collection of tiles T_4 on S_4 such that every cell is covered by exactly three tiles.

We now show that in order for such a triple tiling T_4 of S_4 to exist, n must be a multiple of 3. Indeed, label the $2n \times 2n$ rectangle with "A"s, "B"s, and "C"s as in Figure 3, and note that any band-aid covers exactly one of each letter. As there are $4n^2$ tiles in T_4 and each cell is covered three times, there must be $4n^2/3$ "A"s on the board, and the same number of "B"s and "C"s. Thus, $3 \mid 4n^2$, so $3 \mid n$, as desired. ☐

Solution 2. Suppose the surface is tileable. Label the $3n^2$ cells as shown in Figure 4, and note that the sum of the cells covered by any band-aid is 2. As there are n^2 band-aids in a tiling, the sum of all numbers in the Figure equals $2n^2$, so by symmetry the sum of numbers on one of the three faces is $2n^2/3$. As this must be an integer, the conclusion $3 \mid n$ follows. ☐

Note. To see how Solution 2 is derived from Solution 1, ask yourself the following question: for each cell c in S_3, how many of the four cells in the set $p_4(p_3^{-1}(\{c\}))$ are labelled "A"?



(a) *An illustration of the map* $p_3 : S_{12} \to S_3$*. A copy of* S_{12} *(where the thick dotted edges should be joined together) lies above* S_3*, and the map* p_3 *is given by projecting straight down. The colors illustrate how* S_{12} *can be thought of as four copies of* S_3 *stitched together. Each tile in* T_3 *has exactly four preimages in* S_{12}*, and these preimages form a tiling* T_{12} *of* S_{12}*.*

(b) *The map* $p_4 : S_{12} \to S_4$*, shown in the style of (a). Because there are three cells of* S_{12} *mapping to each cell of* S_4*, the tiling* T_{12} *becomes a triple-tiling of* S_4*.*

Figure 2: *Maps* p_3 *and* p_4*.*

C	A	B	C	A	B	C	A
A	B	C	A	B	C	A	B
B	C	A	B	C	A	B	C
C	A	B	C	A	B	C	A
A	B	C	A	B	C	A	B
B	C	A	B	C	A	B	C
C	A	B	C	A	B	C	A
A	B	C	A	B	C	A	B

Figure 3: *The labeling of the* $2n \times 2n$ *square for the end of Solution 1 (shown for* $n = 4$*). Every band-aid covers one of each letter.*

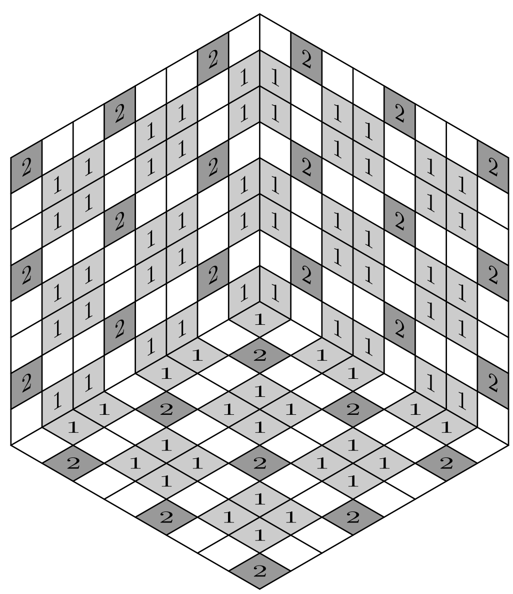


Figure 4: *The labeling used in Solution 2's direct "coloring" argument (shown for* $n = 8$*). Every band-aid covers a sum of* 2*.*