1 A Few Useful Centers

1.1 Symmedian / Lemmoine Point

The Symmedian point $K$ is defined as the isogonal conjugate of the centroid $G$.

**Problem 1.** Show that the symmedian line $AK$ concurs with the tangents to the circumcircle at $B$ and $C$.

**Problem 2.** Point $X$ projects to lines $AB$ and $AC$ at $Y$ and $Z$. Show that $X$ lies on the symmedian line $AK$ if and only if $XY/c = XZ/b$ (where $XY$ and $XZ$ are taken with sign). In particular, the symmedian point is the unique point $P$ whose pedal triangle $RST$ satisfies $PR : PS : PT = a : b : c$.

**Problem 3.** If $D$ is the foot of the $A$-symmedian, show that $BD/DC = (c/b)^2$.

**Problem 4.** The point $K$ is the centroid of its own pedal triangle.

**Problem 5.** Lines $AK$, $BK$, and $CK$ intersect the circumcircle at $A'$, $B'$, and $C'$ respectively (other than $A$, $B$, and $C$). Show that $K$ is the symmedian point of triangle $A'B'C'$.

**Problem 6.** Show that the line connecting the midpoint of $BC$ with the midpoint of the altitude from $A$ passes through $K$.

**Problem 7** (First Lemoine Circle). The the parallels through $K$ are drawn, intercepting the triangle at 6 points. Show that the six points are cyclic with center on $KO$. Also show that the resulting hexagon formed has three equal sides.

**Problem 8** (Second Lemoine Circle). The three antiparallels through $K$ intercept the sides of the triangle in six points. These points lie on a circle with center $K$.

1.2 Gergonne Point and Nagel Point

The Gergonne point $G_e$ is the perspector of $ABC$ with its intouch triangle (which exists by Ceva). The Nagel point $N_a$ is the isotomic conjugate of $G_e$, i.e. the perspector of the extouch triangle.

**Problem 9.** The isogonal conjugate of the Gergonne point is the negative center of homothecy between the incircle and the circumcircle. Likewise, the isogonal conjugate of the Nagel point is the positive center of homothecy of these two circles.
Problem 10. Show that $I$ is the nagel point of the medial triangle. Conclude that $I$, $G$, and $N_a$ are collinear in that order with $IN_a = 3 \cdot IG$.

Problem 11 (Fuhrmann Circle). Let $X$ be the midpoint of arc $BC$ of the circumcircle not containing $A$, and let $X'$ be the reflection of $X$ in side $BC$. Construct $Y'$ and $Z'$ similarly. Then $X'Y'Z'$ lie on a circle with diameter $HN_a$ (known as the Fuhrmann circle). Show also that its center is the reflection of $N_9$ in $I$.

1.3 First and Second Brocard Points

The First Brocard Point $\Omega_1$ is the point so that $\angle \Omega_1 AB = \angle \Omega_1 BC = \angle \Omega_1 CA$. The Second Brocard Point $\Omega_2$ is the isogonal conjugate of $\Omega_1$, i.e. the point so that $\angle \Omega_2 BA = \angle \Omega_2 CB = \angle \Omega_2 AC$. This common angle is often denoted $\omega$.

Problem 12. Give a construction to locate $\Omega_1$ and $\Omega_2$, thus showing that each is unique.

Problem 13. Verify that $\cot(\omega) = \cot(\alpha) + \cot(\beta) + \cot(\gamma)$ and that $\sin^3(\omega) = \sin(\alpha - \omega) \sin(\beta - \omega) \sin(\gamma - \omega)$.

Problem 14. Show that $\omega \leq 30^\circ$. When do we have equality?

Problem 15. Show that the distance from $K$ to side $BC$ is $\frac{1}{2}a \tan \omega$.

Problem 16 (Brocard Circle). Let $B\Omega_1 \cap C\Omega_2 = A_1$, $C\Omega_1 \cap A\Omega_2 = B_1$, $A\Omega_1 \cap B\Omega_2 = C_1$. Triangle $A_1B_1C_1$ is known as the First Brocard Triangle of $\triangle ABC$. Show the following:

- Triangle $A_1B_1C_1$ is inversely similar to $ABC$.
- The circumcircle of $A_1B_1C_1$ contains both brocard points and has diameter $OK$. (This means it is concentric with the First Lemoine Circle.) This is the Brocard Circle.
- $\Omega_1$ and $\Omega_2$ are symmetric about the diameter $KO$.
- If $X$ is the center of the spiral similarity taking segment $CA$ to $AB$, and likewise for $Y$ and $Z$, then $X$, $Y$, and $Z$ also lie on the Brocard Circle.

(Hint: It may help to prove these facts out of order.)

1.4 Fermat Points

If equilateral triangles $BCD$, $CAE$, $ABF$ are constructed on the outside of triangle $ABC$, then $AD$, $BE$, and $CF$ are concurrent at the point known as the first Fermat point. The second Fermat point is the point of concurrency when the triangles are drawn in toward the triangle.

Problem 17. Show that the lines $AD$, $BE$, and $CF$ are in fact concurrent, i.e. the Fermat points exist.

Problem 18. Show that if all angles of $\triangle ABC$ are at most $120^\circ$, then $F_1$ (the first Fermat point) is the point that minimizes the total distance to the three vertices. Where is it if one of the angles is larger than $120^\circ$?
1.5 Isogonal Conjugates (Not a triangle center, but useful anyway!)

**Problem 19.** Reflect $P$ about sides $BC$, $CA$, $AB$ to points $P_a$, $P_b$, $P_c$. Show that $Q$ is the circumcenter of triangle $P_aP_bP_c$.

**Problem 20.** The isogonal conjugate of a point on the circumcircle is at infinity. Pairing this with the previous problem proves what useful result?

**Problem 21.** Define points $P_a$, $P_b$, $P_c$ as in problem 19, and let $DEF$ be the orthic triangle of $\triangle ABC$. Show that $P_aD$, $P_bE$, $P_cF$ are concurrent. Where do they concur? Show that if $P = K$, the symmedian point, this point of concurrency lies on the Euler line.

**Problem 22.** If $AD$ and $BE$ are isogonal cevians, prove that \( \frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2 \).

**Problem 23.** Project $P$ onto $AB$ and $AC$ at $D$ and $E$. Show that $DE \perp AQ$, where $Q$ is the isogonal conjugate of $P$.

1.6 Isotomic Conjugates

**Problem 25.** The isotomic conjugate of an infinite point lies on the Steiner circumellipse, i.e. the ellipse through $A$, $B$, and $C$ that has center $G$.

**Problem 26.** What is the locus of the isotomic conjugate of a point on the circumcircle?

**Problem 27.** Let $P$ and $Q$ be antipodal points on the circumcircle. The lines $PQ^*$ and $QP^*$ joining each of these points to the isotomic conjugate of the other intersect orthogonally on the circumcircle.

**Problem 28.** The isotomic conjugate of $H$ is the symmedian point of the circummedial triangle $DEF$, i.e. the triangle so that $ABC$ is the medial triangle of $DEF$.

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**Problem 29 (Gossard Perspector).** Let $\ell_{\triangle XYZ}$ denote the Euler line of triangle $XYZ$. If $\triangle A$ is the triangle formed by lines $AB$, $AC$, and $\ell_{ABC}$, and similarly for $\triangle B$ and $\triangle C$, then the three lines $\ell_{\triangle A}$, $\ell_{\triangle B}$, $\ell_{\triangle C}$ form a triangle $\triangle DEF$. Show that triangle $DEF$ is perspective with triangle $ABC$ (at a point known as the Gossard Perspector). In fact, show that $\triangle DEF \simeq \triangle ABC$.

**Problem 30 (Schiffler Point).** Show that the euler lines of triangles $ABC$, $ABI$, $BCI$, $CAI$ are concurrent (at the Schiffler point).

**Problem 31 (More Schiffler Point).** Let $A'$ be the midpoint of arc $BC$, and likewise for $B'$ and $C'$. Let $X$ be the nine point center of $A'B'C'$, and let $Y$ be the isogonal conjugate of $X$ through triangle $A'B'C'$. Show that $Y$ is the Schiffler point as above.
Problem 32 (Exeter Point). The median from $A$ intersects the circumcircle again at $A'$, and the tangents to the circumcircle at $B$ and $C$ intersect at $A''$. Similarly define $B'$, $B''$, $C'$, $C''$. Then the lines $A'A''$, $B'B''$, $C'C''$ intersect at the so-called Exeter point, which lies on the Euler line. (It was indeed discovered at Phillips Exeter Academy in 1986.)

Problem 33 (Mittenpunkt). Triangle $DEF$ is the medial triangle of $ABC$. Show that $I_aD$, $I_bE$, $I_cF$ are concurrent (guess where!). Show that this point is the symmedian point of $I_aI_bI_c$.

Problem 34 (Spieker Center). The Spieker center is defined as the incenter of the medial triangle. Show that this is the center of mass of the perimeter of triangle $ABC$.

Problem 35 (Isodynamic Points). Points $D$ and $D'$ are the feet of the internal and external angle bisectors from $A$, respectively, and $C_a$ is the circle with diameter $DD'$. Define the circles $C_b$ and $C_c$ similarly. Then these three circles concur at two points, known as the Isodynamic points $J$ and $J'$. Show that these are the unique points whose pedal triangles are equilateral. (Hint: Show that $J$ and $J'$ are the isogonal conjugates of the Fermat points.)

Problem 36 (deLongchamps Point). The deLongchamps point $L$ is defined as the reflection of $H$ in $O$. Show that the radical center of the circles with centers $A$, $B$, $C$ and radii $a$, $b$, $c$ respectively is the deLongchamps point.

Problem 37 (More deLongchamps Point). Construct an ellipse $E_a$ with foci $B$ and $C$ and passing through $A$. Construct $E_b$ and $E_c$ similarly. Draw the common secant line of each pair of ellipses. Then these lines are concurrent at the deLongchamps point $L$.

3 Problems

Problem 38. Erect squares $BCC_1B_1$, $CAA_2C_2$, and $ABB_3A_3$ outwardly on the sides of $\triangle ABC$. The triangle with sidelines $C_1B_1$, $A_2C_2$, $B_3A_3$ is homothetic with $\triangle ABC$. What is the center of homothecy?

Problem 39 (IMO 1991/5). Prove that inside any triangle $\triangle ABC$, there exist a point $P$ so that one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ has measure at most $30^\circ$.

Problem 40. Let $ABC$ be a triangle, and let the incircle with center $I$ meet $BC$ at $X$. Let $M$ be the midpoint of $BC$. Prove that $MI$ bisects the segment $AX$.

Problem 41. Show that the Nagel point of a triangle lies on the incircle if and only if one of the sides has length $\frac{s}{2}$.

Problem 42. For a given triangle $ABC$, find the point $M$ in the plane such that the sum of the squares of the distances from this point $M$ to the lines $BC$, $CA$, and $AB$ is minimal.

Problem 43. Let $ABC$ be an acute triangle. Points $H$, $D$, and $M$ are the feet of the altitude, angle bisector, and median from $A$, respectively. $S$ and $T$ are the feet of the perpendiculars from $B$ and $C$ respectively to line $AD$. Show that there is a point $P$ on the nine-point circle so that $P$ is equidistant from $H$, $M$, $S$, and $T$. 

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Problem 44. Show that the isodynamic points are inverse in the circumcircle of $ABC$ and that they divide segment $KO$ harmonically.

Problem 45 (Peru TST 2006). In triangle $ABC$, $\omega$ is the circumcircle with center $O$, $\omega_1$ is the circumcircle of $AOC$ with diameter $OQ$. $M$ and $N$ are chosen on $AQ$ and $AC$ such that $ABMN$ is a parallelogram. Prove that $MN$ and $BQ$ intersect on $\omega_1$.

Problem 46 (IMO 2000 Shortlist #21). Let $AH_1$, $BH_2$, and $CH_3$ be the altitudes of an acute triangle $ABC$. The incircle $\omega$ of triangle $ABC$ touches the sides $BC$, $CA$, and $AB$ at $T_1$, $T_2$, and $T_3$ respectively. Consider the symmetric images of the lines $H_1H_2$, $H_2H_3$, and $H_3H_1$ with respect to the lines $T_1T_2$, $T_2T_3$, and $T_3T_1$. Prove that these images form a triangle whose vertices lie on $\omega$.

Problem 47 (USAMO 2001 #2). Let $ABC$ be a triangle and let $\omega$ be its incircle. Denote by $D_1$ and $E_1$ the points where $\omega$ is tangent to sides $BC$ and $AC$, respectively. Denote by $D_2$ and $E_2$ the points on sides $BC$ and $AC$, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by $P$ the point of intersection of segments $AD_2$ and $BE_2$. Circle $\omega$ intersects segment $AD_2$ at two points, the closer of which to the vertex $A$ is denoted by $Q$. Prove that $AQ = D_2P$.

Problem 48. Three circles touch each other externally and all these circles also touch a fixed straight line. Let $A$, $B$, $C$ be the mutual points of contact of these circles. If $\omega$ denotes the Brocard angle of the triangle $ABC$, prove that $\cot \omega = 2$.

Problem 49 (Brazilian MO 2006). An ellipse-shaped billiard table doesn’t have holes. When a ball hits the table border in a point $P$, it follows the symmetric line in respect to the normal line to the ellipse in $P$. Prove that if a ball starts from a point $A$ of the ellipse and, after hitting the table at $B$ and $C$, returns to $A$, then it will return to $B$.

Problem 50 (Baltic Way 2006). Let $ABC$ be a triangle, let $B_1$ be the midpoint of the side $AB$ and $C_1$ the midpoint of the side $AC$. Let $P$ be the point of intersection, other than $A$, of the circumscribed circles around the triangles $ABC_1$ and $AB_1C$. Let $P_1$ be the point of intersection, other than $A$ of the line $AP$ with the circumscribed circle around the triangle $ABC_1$. Prove that $2AP = 3AP_1$.

Problem 51 (Korea 2006). Let $ABC$ be a non-isosceles triangle. The incircle of triangle $ABC$ touches sides $BC$, $CA$, $AB$ at $D$, $E$, $F$. The line $AD$ cuts the incircle at a point $P$ different from $D$. The perpendicular to the line $AD$ at the point $P$ is drawn. The line $EF$ intersects this perpendicular at $Q$. The line $QA$ is drawn. $DE$ and $DF$ cut $QA$ at $R$ and $S$ respectively. Prove that $A$ is the midpoint of $RS$.

Problem 52. Let $O$ be the circumcenter of a triangle $ABC$. The perpendicular bisectors of $AO$, $BO$, $CO$ intersect the lines $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$ respectively. Prove that $A_1$, $B_1$, $C_1$ lie on a line perpendicular to $ON$, where $N$ is the isogonal conjugate to the center of the nine point circle of $\triangle ABC$.

Problem 53 (Romania 2000). Let $ABC$ be an acute triangle, and let $M$ be the midpoint of its side $BC$. Suppose that a point $N$ lies inside the triangle $ABC$ and satisfies $\angle ABN = \angle BAM$ and $\angle ACN = \angle CAM$. Prove that $\angle BAN = \angle CAM$.

Problem 54 (China TST 2005). Let $\omega$ be the circumcircle of acute triangle $ABC$. Two tangents of $\omega$ from $B$ and $C$ intersect at $P$. $AP$ and $BC$ intersect at $D$. Point $E$, $F$ are on $AC$ and $AB$ such that $DE \parallel BA$ and $DF \parallel CA$. 

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1. Prove that $F, B, C, E$ are concyclic.

2. Denote $A_1$ the center of the circle passing through $F, B, C, E$. $B_1, C_1$ are defined similarly. Prove that $AA_1, BB_1, CC_1$ are concurrent.

**Problem 55.** In triangle $ABC$ let $G$ be the centroid, and $A'$ be the intersection point of the $G$-symmedian of $\triangle GBC$ with the circumcircle of $GBC$. Points $B', C'$ are defined like this. Prove that the three circles $AGA'$, $BGB'$, $CGC'$ are coaxal.

**Problem 56.** Reflect $I$ through $BC$ to $A_1$, and similarly for $B_1$ and $C_1$. The lines $AA_1, BB_1, CC_1$ intersect at a point $M$ (prove this!). Show that $IM$ is parallel to the Euler line of $\triangle ABC$. 