Triangle Centers

MOP 2007, Black Group

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1 A Few Useful Centers

1.1 Symmedian / Lemmoine Point

The Symmedian point K is defined as the isogonal conjugate of the centroid G.

Problem 1. Show that the symmedian line AK concurs with the tangents to the circumcircle at B and C.

Problem 2. Point X projects to lines AB and AC at Y and Z. Show that X lies on the symmedian line AK if and only if $XY/c = XZ/b$ (where XY and XZ are taken with sign). In particular, the symmedian point is the unique point P whose pedal triangle RST satisfies $PR : PS : PT = a : b : c$.

Problem 3. If D is the foot of the A-symmedian, show that $\frac{BD}{DC} = \left(\frac{c}{b}\right)^2$.

Problem 4. The point K is the centroid of its own pedal triangle.

Problem 5. Lines AK, BK, and CK intersect the circumcircle at A', B', and C' respectively (other than A, B, and C). Show that K is the symmedian point of triangle $A'B'C'$.

Problem 6. Show that the line connecting the midpoint of BC with the midpoint of the altitude from A passes through K.

Problem 7 (First Lemoine Circle)**.** The the parallels through K are drawn, intercepting the triangle at 6 points. Show that the six points are cyclic with center on KO. Also show that the resulting hexagon formed has three equal sides.

Problem 8 (Second Lemoine Circle)**.** The three antiparallels through K intercept the sides of the triangle in six points. These points lie on a circle with center K.

1.2 Gergonne Point and Nagel Point

The Gergonne point G_e is the perspector of ABC with its intouch triangle (which exists by Ceva). The Nagel point N_a is the isotomic conjugate of G_e, i.e. the perspector of the extouch triangle.

Problem 9. The isogonal conjugate of the Gergonne point is the negative center of homothecy between the incircle and the circumcircle. Likewise, the isogonal conjugate of the Nagel point is the positive center of homothecy of these two circles.

Problem 10. Show that I is the nagel point of the medial triangle. Conclude that I, G, and N_a are collinear in that order with $IN_a = 3 \cdot IG$.

Problem 11 (Fuhrmann Circle)**.** Let X be the midpoint of arc BC of the circumcircle not containing A, and let X' be the reflection of X in side BC. Construct Y' and Z' similarly. Then $X'Y'Z'$ lie on a circle with diameter HN_a (known as the Fuhrmann circle). Show also that its center is the reflection of N_9 in I.

1.3 First and Second Brocard Points

The First Brocard Point Ω_1 is the point so that $\angle\Omega_1 AB = \angle\Omega_1 BC = \Omega_1 CA$. The Second Brocard Point Ω_2 is the isogonal conjugate of Ω_1, i.e. the point so that $\angle\Omega_2 BA = \angle\Omega_2 CB = \angle\Omega_2 AC$. This common angle is often denoted ω.

Problem 12. Give a construction to locate Ω_1 and Ω_2, thus showing that each is unique.

Problem 13. Verify that $\cot(\omega) = \cot(\alpha)+\cot(\beta)+\cot(\gamma)$ and that $\sin^3(\omega) = \sin(\alpha-\omega)\sin(\beta-\omega)\sin(\gamma-\omega)$.

Problem 14. Show that $\omega \leq 30°$. When do we have equality?

Problem 15. Show that the distance from K to side BC is $\frac{1}{2}a\tan\omega$.

Problem 16 (Brocard Circle)**.** Let $B\Omega_1 \cap C\Omega_2 = A_1$, $C\Omega_1 \cap A\Omega_2 = B_1$, $A\Omega_1 \cap B\Omega_2 = C_1$. Triangle $A_1B_1C_1$ is known as the First Brocard Triangle of $\triangle ABC$. Show the following:

- Triangle $A_1B_1C_1$ is inversely similar to ABC.
- The circumcircle of $A_1B_1C_1$ contains both brocard points and has diameter OK. (This means it is concentric with the First Lemoine Circle.) This is the Brocard Circle.
- Ω_1 and Ω_2 are symmetric about the diameter KO.
- If X is the center of the sprial similarity taking segment CA to AB, and likewise for Y and Z, then X, Y, and Z also lie on the Brocard Circle.

(Hint: It may help to prove these facts out of order.)

1.4 Fermat Points

If equilateral triangles BCD, CAE, ABF are constructed on the outside of triangle ABC, then AD, BE, and CF are concurrent at the point known as the first Fermat point. The second Fermat point is the point of concurrency when the triangles are drawn in toward the triangle.

Problem 17. Show that the lines AD, BE, and CF are in fact concurrent, i.e. the Fermat points exist.

Problem 18. Show that if all angles of $\triangle ABC$ are at most $120°$, then F_1 (the first Fermat point) is the point that minimizes the total distance to the three vertices. Where is it if one of the angles is larger than $120°$?

1.5 Isogonal Conjugates (Not a triangle center, but useful anyway!)

Problem 19. Reflect P about sides BC, CA, AB to points P_a, P_b, P_c. Show that Q is the circumcenter of triangle $P_aP_bP_c$.

Problem 20. The isogonal conjugate of a point on the circumcircle is at infinity. Pairing this with the previous problem proves what useful result?

Problem 21. Define points P_a, P_b, P_c as in problem 19, and let DEF be the orthic triangle of $\triangle ABC$. Show that P_aD, P_bE, P_cF are concurrent. Where do they concur? Show that if $P = K$, the symmedian point, this point of concurrency lies on the Euler line.

Problem 22. If AD and BE are isogonal cevians, prove that $\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2$.

Problem 23. Project P onto AB and AC at D and E. Show that $DE \perp AQ$, where Q is the isogonal conjugate of P.

Problem 24. Lines ℓ_a and ℓ'_a are isogonal lines through vertex A, and the pairs of lines ℓ_b, ℓ'_b and ℓ_c, ℓ'_c are chosen similarly. Let $X = \ell_b \cap \ell'_c$, $Y = \ell_c \cap \ell'_a$, $Z = \ell_a \cap \ell'_b$. Show that AX, BY, CZ are concurrent. Also, let $X' = \ell'_b \cap \ell_c$, $Y' = \ell'_c \cap \ell_a$, $Z' = \ell'_a \cap \ell_b$, and show that lines XX', YY', and ZZ' are concurrent.

1.6 Isotomic Conjugates

Problem 25. The isotomic conjugate of an infinite point lies on the Steiner circumellipse, i.e. the ellipse through A, B, and C that has center G.

Problem 26. What is the locus of the isotomic conjugate of a point on the circumcircle?

Problem 27. Let P and Q be antipodal points on the circumcircle. The lines $PQ^\bullet$ and $QP^\bullet$ joining each of these points to the isotomic conjugate of the other intersect orthogonally on the circumcircle.

Problem 28. The isotomic conjugate of H is the symmedian point of the circummedial triangle DEF, i.e. the triangle so that ABC is the medial triangle of DEF.

2 A Few Useless Centers

Problem 29 (Gossard Perspector)**.** Let $\ell_{\triangle XYZ}$ denote the Euler line of triangle XYZ. If $\triangle A$ is the triangle formed by lines AB, AC, and ℓ_{ABC}, and similarly for $\triangle B$ and $\triangle C$, then the three lines $\ell_{\triangle A}$, $\ell_{\triangle B}$, $\ell_{\triangle C}$ form a triangle $\triangle DEF$. Show that triangle DEF is perspective with triangle ABC (at a point known as the Gossard Perspector). In fact, show that $\triangle DEF \simeq \triangle ABC$.

Problem 30 (Schiffler Point)**.** Show that the euler lines of triangles ABC, ABI, BCI, CAI are concurrent (at the Schiffler point).

Problem 31 (More Schiffler Point)**.** Let A' be the midpoint of arc BC, and likewise for B' and C'. Let X be the nine point center of $A'B'C'$, and let Y be the isogonal conjugate of X through triangle $A'B'C'$. Show that Y is the Schiffler point as above.

Problem 32 (Exeter Point)**.** The median from A intersects the circumcircle again at A', and the tangents to the circumcircle at B and C intersect at A''. Similarly define B', B'', C', C''. Then the lines $A'A''$, $B'B''$, $C'C''$ intersect at the so-called Exeter point, which lies on the Euler line. (It was indeed discovered at Phillips Exeter Academy in 1986.)

Problem 33 (Mittenpunkt)**.** Triangle DEF is the medial triangle of ABC. Show that I_aD, I_bE, I_cF are concurrent (guess where!). Show that this point is the symmedian point of $I_aI_bI_c$.

Problem 34 (Spieker Center)**.** The Spieker center is defined as the incenter of the medial triangle. Show that this is the center of mass of the perimeter of triangle ABC.

Problem 35 (Isodynamic Points)**.** Points D and D' are the feet of the internal and external angle bisectors from A, respectively, and $\mathcal{C}_a$ is the circle with diameter DD'. Define the circles $\mathcal{C}_b$ and $\mathcal{C}_c$ similarly. Then these three circles concur at two points, known as the Isodynamic points J and J'. Show that these are the unique points whose pedal triangles are equilateral. (Hint: Show that J and J' are the isogonal conjugates of the Fermat points.)

Problem 36 (deLongchamps Point)**.** The deLongchamps point L is defined as the reflection of H in O. Show that the radical center of the circles with centers A, B, C and radii a, b, c respectively is the deLongchamps point.

Problem 37 (More deLongchamps Point)**.** Construct an ellipse $\mathcal{E}_a$ with foci B and C and passing through A. Construct $\mathcal{E}_b$ and $\mathcal{E}_c$ similarly. Draw the common secant line of each pair of ellipses. Then these lines are concurrent at the deLongchamps point L.

3 Problems

Problem 38. Erect squares BCC_1B_1, CAA_2C_2, and ABB_3A_3 outwardly on the sides of $\triangle ABC$. The triangle with sidelines C_1B_1, A_2C_2, B_3A_3 is homothetic with $\triangle ABC$. What is the center of homothecy?

Problem 39 (IMO 1991/5)**.** Prove that inside any triangle $\triangle ABC$, there exist a point P so that one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ has measure at most $30°$.

Problem 40. Let ABC be a triangle, and let the incircle with center I meet BC at X. Let M be the midpoint of BC. Prove that MI bisects the segment AX.

Problem 41. Show that the Nagel point of a triangle lies on the incircle if and only if one of the sides has length $\frac{s}{2}$.

Problem 42. For a given triangle ABC, find the point M in the plane such that the sum of the squares of the distances from this point M to the lines BC, CA, and AB is minimal.

Problem 43. Let ABC be an acute triangle. Points H, D, and M are the feet of the altitude, angle bisector, and median from A, respectively. S and T are the feet of the perpendiculars from B and C respectively to line AD. Show that there is a point P on the nine-point circle so that P is equidistant from H, M, S, and T.

Problem 44. Show that the isodynamic points are inverse in the circumcircle of ABC and that they divide segment KO harmonically.

Problem 45 (Peru TST 2006)**.** In triangle ABC, ω is the circumcircle with center O, ω_1 is the circumcircle of AOC with diameter OQ. M and N are chosen on AQ and AC such that $ABMN$ is a parallelogram. Prove that MN and BQ intersect on ω_1.

Problem 46 (IMO 2000 Shortlist #21)**.** Let AH_1, BH_2, and CH_3 be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA, and AB at T_1, T_2, and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3, and T_3T_1. Prove that htese images form a triangle whose vertices lie on ω.

Problem 47 (USAMO 2001 #2)**.** Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of whihc to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

Problem 48. Three circles touch each other externally and all these circles also touch a fixed straight line. Let A, B, C be the mutual points of contact of these circles. If ω denotes the Brocard angle of the triangle ABC, prove that $\cot \omega = 2$.

Problem 49 (Brazilian MO 2006)**.** An ellipse-shaped billiard table doesn't have holes. When a ball hits the table border in a point P, it follows the symmetric line in respect to the normal line to the ellipse in P. Prove that if a ball starts from a point A of the ellipse and, after hitting the table at B and C, returns to A, then it will return to B.

Problem 50 (Baltic Way 2006)**.** Let ABC be a triangle, let B_1 be the midpoint of the side AB and C_1 the midpoint of the side AC. Let P be the point of intersection, other than A, of the circumscribed circles around the triangles ABC_1 and AB_1C. Let P_1 be the point of intersection, other than A of the line AP with the circumscribed circle around the triangle AB_1C_1. Prove that $2AP = 3AP_1$.

Problem 51 (Korea 2006)**.** Let ABC be a non-isosceles triangle. The incircle of triangle ABC touches sides BC, CA, AB at D, E, F. The line AD cuts the incircle at a point P different from D. The perpendicular to the line AD at the point P is drawn. The line EF intersects this perpendicular at Q. The line QA is dran. DE and DF cut QA at R and S respectively. Prove that A is the midpoint of RS.

Problem 52. Let O be the circumcenter of a triangle ABC. The perpendicular bisectors of AO, BO, CO intersects the lines BC, CA, AB at A_1, B_1, C_1 respectively. Prove that A_1, B_1, C_1 lie on a line perpendicular to ON, where N is the isogonal conjugate to the center of the nine point circle of $\triangle ABC$.

Problem 53 (Romania 2000)**.** Let ABC be an acute triangle, and let M be the midpoint of its side BC. Suppose that a point N lies inside the triangle ABC and satisfies $\angle ABN = \angle BAM$ and $\angle ACN = \angle CAM$. Prove that $\angle BAN = \angle CAM$.

Problem 54 (China TST 2005)**.** Let ω be the circumcircle of acute triangle ABC. Two tangents of ω from B and C intersect at P. AP and BC intersect at D. Point E, F are on AC and AB such that $DE \parallel BA$ and $DF \parallel CA$.

1. Prove that F, B, C, E are concyclic.

2. Denote A_1 the center of the circle passing through F, B, C, E. B_1, C_1 are defined similarly. Prove that AA_1, BB_1, CC_1 are concurrent.

Problem 55. In triangle ABC let G be the centroid, and A' be the intersection point of the G-symmedian of $\triangle GBC$ with the circumcircle of GBC. Points B', C' are defined like this. Prove that the three circles AGA', BGB', CGC' are coaxal.

Problem 56. Reflect I through BC to A_1, and similarly for B_1 and C_1. The lines AA_1, BB_1, and CC_1 intersect at a point M (prove this!). Show that IM is parallel to the Euler line of $\triangle ABC$.