

A Topologically Convex Vertex-Ununfoldable Polyhedron

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Vertex-Unfolding Fundamentals

- Vertex-unfolding is like edge-unfolding
 - ▶ Cut some edges
 - ▶ Leave it connected
 - ▶ Fold it flat

Vertex-Unfolding Fundamentals

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 - ▶ Cut some edges: **Cut *all* edges!**
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Vertex-Unfolding Fundamentals

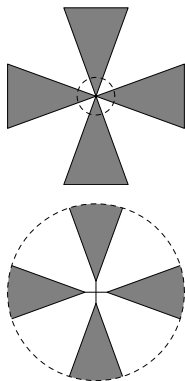
- Vertex-unfolding is like edge-unfolding
 - ▶ Cut some edges: Cut *all* edges!
 - ▶ Leave it connected *with vertex hinges*
 - ▶ Fold it flat

Vertex-Unfolding Fundamentals

- Vertex-unfolding is like edge-unfolding
 - ▶ Cut some edges: Cut *all* edges!
 - ▶ Leave it connected with *vertex hinges*
 - ▶ Fold it flat: *folds like a hinged figure*

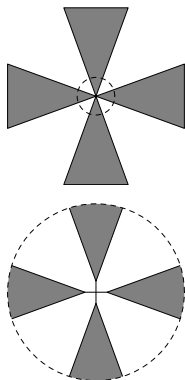
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- Crossing hinges are not allowed.



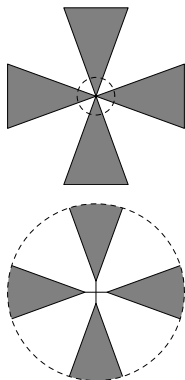
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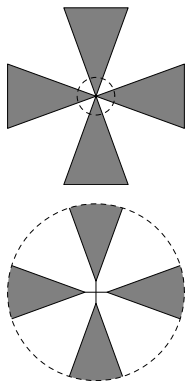


Open Question

Can every convex polyhedron be edge-unfolded?

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Open Question (Weaker)

Can every convex polyhedron be **vertex**-unfolded?

Previous Work: Positive Results

Theorem (DEE+02)

Any triangulated manifold can be vertex-unfolded.

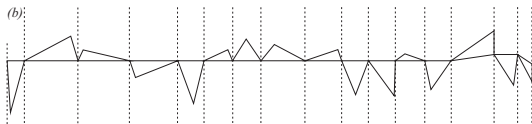
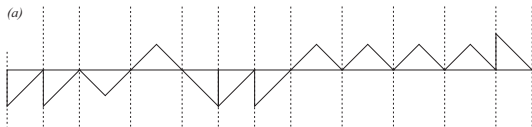
- So the Witch's Hat Tetrahedron has a vertex unfolding (but no edge unfolding).

Previous Work: Positive Results

Theorem (DEE+02)

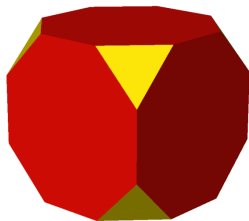
*Any triangulated manifold can be vertex-unfolded **into a chain**.*

- So the Witch's Hat Tetrahedron has a vertex unfolding (but no edge unfolding).



Previous Work: Negative Results

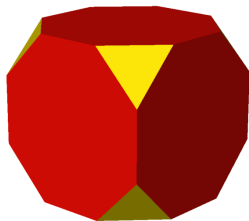
Not every convex polyhedron has a
chain vertex-unfolding [DEE+02]:



[Image source: Wikipedia]

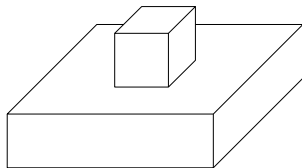
Previous Work: Negative Results

Not every convex polyhedron has a **chain** vertex-unfolding [DEE+02]:



[Image source: Wikipedia]

Not every polyhedron has a vertex-unfolding [BDDLOORW, CCCG '98]:



Local Obstructions to Vertex-Unfolding

Suppose we have two vertices v_1, v_2 of different polygons with angles α_1, α_2 respectively.

Observation 1

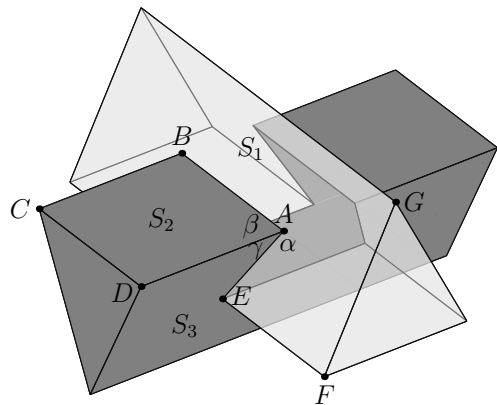
If $\alpha_1 + \alpha_2 > 360^\circ$, these vertices cannot be hinged in the plane without overlap.

Observation 2

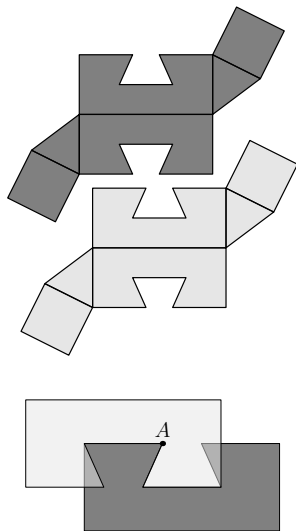
If $\alpha_1 + \alpha_2 = 360^\circ$, and the polygons are hinged at these vertices without overlap, then they must be oriented to exactly cover the 360° surrounding the hinge.

A New Vertex-Ununfoldable Polyhedron

Topologically Convex

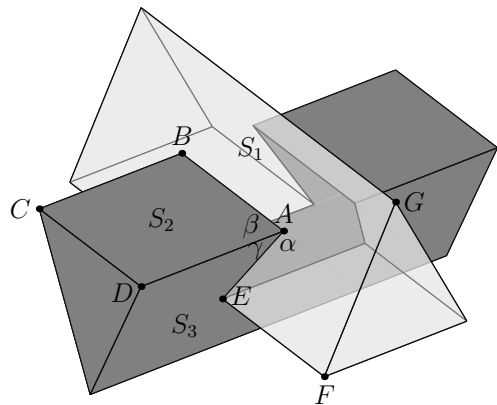


Polyhedron *P*

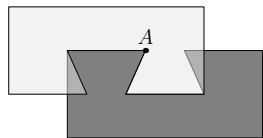
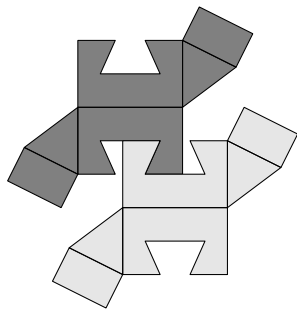


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Topologically Convex

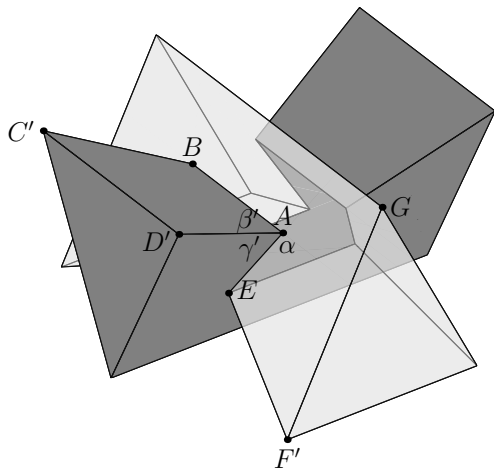


Polyhedron P



A More Local Example

Topologically Convex



Polyhedron P'

Looking Forward

Which families of polyhedra have vertex unfoldings?

- Not always:
 - ▶ All Polyhedra
 - ▶ Topologically convex (and star-shaped)
- Open:
 - ▶ Convex faces (and topologically convex)
 - ▶ Convex
- Always:
 - ▶ Triangulated

Complexity of vertex-unfolding?