FEATURE

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MATHEMATICAL MINUTIAE

i Has This Funny Property

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Recall that the **derivative** of a real-valued function $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g'(z) = \lim_{h \to 0} \frac{g(z+h) - g(z)}{h}.$$

While this definition implies continuity, woe be unto the **real analyst**[1] who tries to differentiate the function

$$h_1(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x \geq 0 \end{cases}$$

twice, even though he may do so once. Indeed, many a differentiable, real-valued function is not twice differentiable.

We define the **complex derivative** of a function $f : \mathbb{C} \to \mathbb{C}$ as

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

where h ranges over complex numbers (if this limit exists). Although this definition looks similar to that of the real derivative, the real and complex derivatives are wildly different beasts. For example, we have the following, which shows that—unlike real-differentiable functions—complex-differentiable functions are always multiply differentiable.

Theorem 1 (Cauchy's Integral Formula [SS, Cor. 4.2] [La, Ch. III Thm. 7.7])**.** *Any complex function $f : \mathbb{C} \to \mathbb{C}$ that is differentiable near $z_0 \in \mathbb{C}$ is infinitely differentiable there. Furthermore,*

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[1] real a·na·lyst, (ˈriːəl ˈænəlɪst), *noun*. (1) A mathematician who studies the analytic properties of the real numbers. (2) An analyst who is not fake.

we may explicitly calculate these values:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz,$$

where γ is any sufficiently small loop around z_0.

Not only is a complex differentiable function twice differentiable, it is actually **smooth**! Furthermore, it has a convenient power series representation:

Theorem 2 ([SS, Thm. 4.4] [La, Ch. IV Thm. 7.3])**.** *If f is complex-differentiable near z_0, then f is **analytic** near z_0, i.e. it has a power series expansion*

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

in an open neighborhood of z_0.

In light of Theorem 1, we can take Theorem 2 even further, explicitly calculating the power series coefficients: $c_k = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-z_0)^{n+1}}$. This is a far cry from the real-analytic case, where even an infinitely differentiable function may not have a power series expansion. The function

$$h_2(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0. \end{cases}$$

is a classic example of a smooth function $h_2 : \mathbb{R} \to \mathbb{R}$ with no power series expansion at $x = 0$. (See if you can prove this!)

For any differentiable function $g : \mathbb{R} \to \mathbb{R}$, the image set $g(\mathbb{R})$ is simply an interval.[2] With our trusty i, however, we can see far more about the shape of the image:

Theorem 3 (Liouville's Theorem [SS, Cor. 4.5] [La, Ch. III, Thm. 7.5])**.** *If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function such that the image $f(\mathbb{C})$ is bounded, then f is constant.*

Whereas in the real case we could only describe the image of a function as an interval, in the complex case we know instead that the image is bounded if and only if it is a single point. We furthermore have control over the images of complex analytic functions with unbounded images:

Theorem 4 (Picard's Little Theorem [SS, Exer. 6.11] [La, Ch. XII, Thm. 2.8])**.** *If an analytic function $f : \mathbb{C} \to \mathbb{C}$ is nonconstant, then its image omits at most one value. That is, $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{p\}$ for some $p \in \mathbb{C}$.*

So much power in one little i!

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[2] Incidentally, it may be *any* interval—closed, open, or half-open. In fact, this works even if g is only known to be continuous. (Prove it!)

[3] In addition to [La] and [SS], the authors recommend [Al] and [Re], both of which were used as course texts in Nicoara's Mathematics 213a.

References

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